

# Slope Rotatable Central Composite Designs of Second Type

## ABSTRACT

Central composite design (CCD) is the most commonly used fractional factorial design used in the response surface model. Kim (2002) proposed second order rotatable designs (SORD) of second type using CCD, in which the positions of axial points are indicated by two numbers  $(a_1, a_2)$ . Kim and Ko (2004) introduced second order slope rotatable designs (SOSRD) of second type using CCD, in which the positions of axial points are indicated by two numbers  $(a_1, a_2)$ . In this paper, second order slope rotatable central composite designs of second type with  $2 \leq n_a \leq 4$  (where  $n_a$  denotes the number of replications of axial points) are suggested for  $2 \leq v \leq 17$  ( $v$ -stands for number of factors). It is observed that the value of level  $a_2$  (taking  $a_1 = 1$ ) for the axial points in CCD required for slope rotatability for second type is appreciably larger than the value required for SORD of second type using CCD. And also noted that if we replicate axial points ( $n_a$ ) in SOSRD of second type using CCD then the value of  $a_2$  (taking  $a_1 = 1$ ) is approximately nearer to SORD value  $a_2$  of second type using CCD.

*Keywords: Response surface designs, Second order slope rotatable designs, Slope rotatable central composite designs, Second order slope rotatable designs of second type.*

## 1. INTRODUCTION

Response surface design is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence a dependent variable. The property of rotatability was proposed by Box and Hunter (1957) for response surface designs and constructed second order rotatable central composite designs (CCD). Das and Narasimham (1962) constructed second order rotatable designs (SORD) using balanced incomplete block designs (BIBD). Draper and Guttman (1988) suggested an index of rotatability. Khuri (1988) introduced measure of rotatability for response surface designs. Draper and Pukelshein (1990) developed another look at rotatability. Park et al. (1993) suggested measure of rotatability for second order response surface designs. Kim (2002) introduced extended central composite designs with the axial points indicated by two numbers. Victorbabu and Vasundaradevi (2005) suggested modified second order response designs, rotatable designs using BIBD. Victorbabu and Surekha (2013) studied on measure of rotatability for second order response surface designs using BIBD. Jyostna et al. (2021) suggested measure of modified rotatability for second order response surface designs using CCD. Jyostna and Victorbabu (2021) studied measure of modified

rotatability for second degree polynomial using BIBD. Chiranjeevi et al. (2021) developed SORD of second type using CCD. Chiranjeevi and Victorbabu (2021) studied SORD of second type using BIBD.

Hader and Park (1978) introduced slope rotatable central composite designs (SRCCD). Victorbabu and Narasimham (1991a) constructed second order slope rotatable designs (SOSRD) using BIBD. Victorbabu and Narasimham (1991b) studied SOSRD through a pair of incomplete block designs. Park and Kim (1992) developed measure of slope rotatability for second order response surface experimental designs. Victorbabu (2005, 2006) introduced modified SOSRD using CCD and BIBD respectively. Victorbabu (2007) suggested a review on SOSRD. Victorbabu and Surekha (2011, 2012) studied measure of SOSRD using CCD and BIBD respectively. Rajyalakshmi and Victorbabu (2014) studied an empirical study on robustness of SOSRD using symmetrical unequal block arrangements with two unequal block sizes. Rajyalakshmi and Victorbabu (2019) constructed SOSRD under tri-diagonal correlated structure of errors using BIBD. Rajyalakshmi et al. (2020) studied SOSRD under intra-class correlated errors using pairwise balanced designs. Sulochana and Victorbabu (2020a) studied SOSRD under intra-class correlated structure of errors using partially balanced incomplete block type designs. Sulochana and Victorbabu (2020b) studied SOSRD under tri-diagonal correlation structure of errors using a pair of incomplete block designs. Victorbabu and Jyostna (2021) studied measure of modified slope rotatability for second order response surface designs. Specifically, Kim and Ko (2004) introduced slope rotatability for CCD of second type for  $2 \leq v \leq 5$  ( $v$ -stands for number of factors) by taking  $n_a=1$  (where  $n_a$  denotes the number of replications of axial points), in which the positions of axial points are indicated by two numbers  $(a_1, a_2)$ . Ravikumar and Victorbabu (2022) extended the work of Kim and Ko (2004) and developed SOSRD of second type using CCD for  $6 \leq v \leq 17$  by taking  $n_a=1$ . Victorbabu and Ravikumar (2022) developed SOSRD of second type using BIBD.

In this paper an attempt is made to study SRCCD of second type with  $2 \leq n_a \leq 4$  (where  $n_a$  denotes the number of replications of axial points) for  $2 \leq v \leq 17$ . It is observed that the value of level  $a_2$  (taking  $a_1 = 1$ ) required for SOSRD of second type using CCD is appreciably larger than the value required for SORD of second type using CCD, and also noted that if we replicate axial points ( $n_a$ ) in SOSRD of second type using CCD then the value of  $a_2$  is approximately nearer to SORD of second type using CCD.

## 2. CONDITIONS FOR SECOND ORDER SLOPE ROTATABLE DESIGNS

A general second order response surface design  $D = ((X_{iu}))$  for fitting

$$Y_u = \beta_0 + \sum_{i=1}^v \beta_i X_{iu} + \sum_{i=1}^v \beta_{ii} X_{iu}^2 + \sum_{i=1}^v \sum_{j=1}^v \beta_{ij} X_{iu} X_{ju} + e_u$$

(2.1) where  $X_{iu}$  denotes the level of the  $i^{\text{th}}$  factor ( $i=1,2,\dots,v$ ) in the  $u^{\text{th}}$  run ( $u=1,2,\dots,N$ ) of the experiment and  $e_u$ 's are uncorrelated random errors with mean zero and variance  $\sigma^2$ . Then  $D$  is said to be SOSRD if the variance of the estimate of the first order partial derivative of  $Y(X_1, X_2, \dots, X_v)$  with respect to

each of independent variable  $X_i$  is only a function of the distance  $\left(d^2 = \sum_{i=1}^v X_i^2\right)$  of the point  $(X_1, X_2, \dots, X_v)$  from the origin (centre) of the design.

The general conditions for second order slope rotatable designs are as follows [cf. Box and Hunter (1957), Hader and Park (1978) and Victorbabu and Narasimham (1991a)].

All odd order moments are zero. In other words when at least one odd power  $X$ 's equal to zero. i.e;

$$\begin{aligned}
 \text{A. } & \sum X_{iu} = 0, \sum X_{iu} X_{ju} = 0, \sum X_{iu} X_{ju}^2 = 0, \sum X_{iu} X_{ju} X_{ku} = 0, \\
 & \sum X_{iu}^3 = 0, \sum X_{iu} X_{ju}^3 = 0, \sum X_{iu} X_{ju} X_{ku}^2 = 0, \sum X_{iu} X_{ju} X_{ku} X_{lu} = 0, \text{etc. for } i \neq j \neq k \neq l; \\
 \text{B. (i)} & \sum X_{iu}^2 = \text{constant} = N\lambda_2 \\
 & \text{(ii) } \sum X_{iu}^4 = \text{constant} = cN\lambda_4, \text{ for all } i \\
 \text{C. } & \sum X_{iu}^2 X_{ju}^2 = \text{constant} = N\lambda_4, \text{ for all } i \neq j \tag{2.2}
 \end{aligned}$$

where  $c$ ,  $\lambda_2$  and  $\lambda_4$  are constants.

The variances and covariances of the estimated parameters are

$$\begin{aligned}
 V(\hat{\beta}_0) &= \frac{\lambda_4(c+v-1)\sigma^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]} \\
 V(\hat{\beta}_i) &= \frac{\sigma^2}{N\lambda_2} \\
 V(\hat{\beta}_{ij}) &= \frac{\sigma^2}{N\lambda_4} \\
 V(\hat{\beta}_{ii}) &= \frac{\sigma^2}{(c-1)N\lambda_4} \left[ \frac{\lambda_4(c+v-2)-(v-1)\lambda_2^2}{\lambda_4(c+v-1)-v\lambda_2^2} \right] \\
 \text{Cov}(\hat{\beta}_0, \hat{\beta}_{ii}) &= \frac{-\lambda_2\sigma^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]} \\
 \text{Cov}(\hat{\beta}_{ii}, \hat{\beta}_{ij}) &= \frac{(\lambda_2^2 - \lambda_4)\sigma^2}{(c-1)N\lambda_4[\lambda_4(c+v-1)-v\lambda_2^2]} \text{ and other covariances vanish.} \tag{2.3}
 \end{aligned}$$

An inspection of the  $V(\hat{\beta}_0)$  shows that a necessary condition for the existence of a non singular second order design is

$$\text{D. } \frac{\lambda_4}{\lambda_2^2} > \frac{v}{c+v-1} \quad (\text{Non-singularity condition}) \tag{2.4}$$

For the second order model (2.1), we have

$$\frac{\partial \hat{Y}}{\partial X_i} = \hat{\beta}_i + 2\hat{\beta}_{ii}X_{iu} + \sum_{j \neq i} \hat{\beta}_{ij}X_{ju} \tag{2.5}$$

$$V\left(\frac{\partial \hat{Y}}{\partial X_i}\right) = V(\hat{\beta}_i) + 4X_{iu}^2 V(\hat{\beta}_{ii}) + \sum_{j \neq i} X_{ju}^2 V(\hat{\beta}_{ij})$$

(2.6) The condition for R.H.S of the equation (2.6) to be a function of  $d^2 = \sum_{i=1}^v X_i^2$  alone (for slope

rotatability) is  $4V(\hat{\beta}_{ii}) = V(\hat{\beta}_{ij})$  [cf. Hader and Park (1978)]

(2.7)

On simplification of (2.7), using (2.3) we get,

$$E. \lambda_4 [v(5-c)-(c-3)^2] + \lambda_2^2 [v(c-5)+4] = 0 \quad \text{[cf. Victorbabu and Narasimham (1991a)]} \quad (2.8)$$

Therefore A, B, C of (2.2), (2.4) and (2.8) give a set of conditions for slope rotatability in any general second order response design.

### 3. SECOND ORDER ROTATABLE DESIGNS OF SECOND TYPE USING CENTRAL COMPOSITE DESIGNS

Kim (2002) developed second type of rotatable central composite designs (CCD) in which the positions of axial points are indicated by two numbers  $(a_1, a_2)$  for  $2 \leq v \leq 8$ . Chiranjeevi et al. (2021) extended the results of Kim (2002) and developed SORD of second type using CCD for  $9 \leq v \leq 17$ . Chiranjeevi and Victorbabu (2021) studied SORD of second type using BIBD.

The design plan of SORD of second type using CCD in which the positions of the axial points are indicated by two numbers  $a_1$  and  $a_2$  ( $a_2 \geq a_1 > 0$ ). The CCD are constructed by adding suitable fractional combinations to those obtained from  $\frac{1}{2^p} \times 2^v$  fractional factorial design, (here  $2^{t(v)} = \frac{1}{2^p} \times 2^v$  denotes a suitable fractional replicate of  $2^v$ ), in which no interaction with less than five factors are confounded. In coded form CCD has the points of  $2^v$  ( $2^{t(v)}$ ) factorial with coordinates  $(\pm 1, \pm 1, \dots, \pm 1)$  and  $4v$  axial points with coordinates  $(\pm a_1, 0, \dots, 0), (0, \pm a_1, \dots, 0), \dots, (0, 0, \dots, \pm a_1); (\pm a_2, 0, \dots, 0), (0, \pm a_2, \dots, 0), \dots, (0, 0, \dots, \pm a_2)$  and if necessary  $n_0$  central points may be replicated. Thus the total number of experimental points  $N = 2^{t(v)} + 4v + n_0$ .

For the design points generated from CCD, simple symmetry conditions A, B and C of equation (2.2) are true. Condition (A) of equation (2.2) is true obviously, condition (B) and (C) are true as follows.

$$B. (i) \sum X_{iu}^2 = 2^{t(v)} + 2a_1^2 + 2a_2^2 = N\lambda_2$$

$$(ii) \sum X_{iu}^4 = 2^{t(v)} + 2a_1^4 + 2a_2^4 = 3N\lambda_4$$

$$C. \sum X_{iu}^2 X_{ju}^2 = 2^{t(v)} = N\lambda_4 \quad (3.1)$$

From B(ii) and C of equation (3.1), we get

$$2^{t(v)} + 2a_1^4 + 2a_2^4 = 3(2^{t(v)}) \Rightarrow a_1^4 + a_2^4 = 2^{t(v)} \quad (3.2)$$

**Example (3.1):**

We illustrate the method of SORD of second type using CCD for  $v=6$ . The design points  $(\pm 1, \pm 1, \dots, \pm 1) 2^{(6)} U(\pm a_1, 0, \dots, 0) 2^1 U(\pm a_2, 0, \dots, 0) 2^1 U(n_0=1)$  will give a SORD of second type in  $N=57$  design points with  $n_a=1$ .

For the design points generated from SORD of second type using CCD, simple symmetry conditions A of equation (2.2) are true.

Here B and C of equation (3.1) are

$$\begin{aligned}
 \text{B. (i)} \quad & \sum X_{iu}^2 = 32 + 2a_1^2 + 2a_2^2 = N\lambda_2 \\
 \text{(ii)} \quad & \sum X_{iu}^4 = 32 + 2a_1^4 + 2a_2^4 = 3N\lambda_4 \\
 \text{C.} \quad & \sum X_{iu}^2 X_{ju}^2 = 32 = N\lambda_4
 \end{aligned} \tag{3.3}$$

From B (ii) and C of equation (3.3), we get  $a_2 = 2.3596$  (taking  $a_1 = 1$ ).

#### 4. A NEW METHOD OF SLOPE ROTATABLE CENTRAL COMPOSITE DESIGNS OF SECOND TYPE

Kim and Ko (2004) introduced slope rotatability of second type using CCD for  $2 \leq v \leq 5$  by taking  $n_a=1$ . Ravikumar and Victorbabu (2022) developed SOSRD of second type using CCD for  $6 \leq v \leq 17$  with  $n_a=1$ .

The design plan of SOSRD of second type using CCD in which the positions of the axial points are indicated by two numbers  $a_1$  and  $a_2$  ( $a_2 \geq a_1 > 0$ ). The CCD are constructed by adding suitable fractional combinations to those obtained from  $\frac{1}{2^p} \times 2^v$  fractional factorial design, (here  $2^{(v)} = \frac{1}{2^p} \times 2^v$  denotes a suitable fractional replicate of  $2^v$ ), in which no interaction with less than five factors are confounded. In coded form CCD has the points of  $2^v$  ( $2^{(v)}$ ) factorial with coordinates  $(\pm 1, \pm 1, \dots, \pm 1)$  and  $4v$  axial points are replicated  $n_a$  times with coordinates  $(\pm a_1, 0, \dots, 0), (0, \pm a_1, \dots, 0), \dots, (0, 0, \dots, \pm a_1); (\pm a_2, 0, \dots, 0), (0, \pm a_2, \dots, 0), \dots, (0, 0, \dots, \pm a_2)$  and if necessary  $n_0$  central points may be replicated. Thus the total number of experimental points  $N = 2^{(v)} + 4n_a v + n_0$ .

For CCD simple symmetry conditions A, B and C of equation (2.2) are true for any  $n_a$ . Condition (A) of equation (2.2) is true obviously, condition (B) and (C) are true as follow:

$$\begin{aligned}
 \text{B. (i)} \quad & \sum X_{iu}^2 = 2^{(v)} + 2n_a a_1^2 + 2n_a a_2^2 = N\lambda_2 \\
 \text{(ii)} \quad & \sum X_{iu}^4 = 2^{(v)} + 2n_a a_1^4 + 2n_a a_2^4 = cN\lambda_4 \\
 \text{C.} \quad & \sum X_{iu}^2 X_{ju}^2 = 2^{(v)} = N\lambda_4
 \end{aligned} \tag{4.1}$$

From B(ii) and C of equation (4.1), we have  $c = \frac{2^{t(v)} + 2n_a a_1^4 + 2n_a a_2^4}{2^{t(v)}}$

Substituting  $\lambda_2$ ,  $\lambda_4$  and c in equation (2.8) and on simplification, we get the following biquadratic equation

$$\begin{aligned} & \left[ 2N(n_a)^2 - 4v(n_a)^3 \right] (a_1^8 + a_2^8) - 8v(n_a)^3 (a_1^6 a_2^2 + a_1^2 a_2^6) + 4 \left[ N(n_a)^2 - 2v(n_a)^3 \right] a_1^4 a_2^4 - 2^{t(v)+2} v(n_a)^2 \left[ a_1^6 + a_1^4 a_2^2 + a_1^2 a_2^4 + a_2^6 \right] \\ & - 2^{t(v)} \left[ v(n_a) 2^{t(v)} + 8(1-v)(n_a)^2 + N(4-v)(n_a) \right] (a_1^4 + a_2^4) + 2^{t(v)+4} (v-1)(n_a)^2 a_1^2 a_2^2 + 2^{2t(v)+3} (v-1)(n_a) (a_1^2 + a_2^2) \\ & - 2^{2t(v)+1} (1-v) (2^{t(v)} - N) = 0 \end{aligned} \quad (4.2)$$

If at least one positive real root exists for the above equation (4.2), then the design exists. Given the values of v,  $n_a$  and  $n_0$  there are countless combinations of  $a_1$  and  $a_2$  that satisfy the equation (4.2).

#### Example (4.1):

We illustrate the new method SOSRD of second type using CCD for v=6. The design points  $(\pm 1, \pm 1, \dots, \pm 1) 2^{t(6)} U_{n_a} (\pm a_1, 0, \dots, 0) 2^1 U_{n_a} (\pm a_2, 0, \dots, 0) 2^1 U_{(n_0=26)}$  will give a SOSRD of second type in N=106 design points with  $n_a=2$ ,  $a_1=1$ .

For the design points generated from SOSRD of second type using CCD, simple symmetry condition A of equation (2.2) are true for any  $n_a$ .

Here B and C of equation (4.1) are

$$B. (i) \sum X_{iu}^2 = 32 + 4a_1^2 + 4a_2^2 = N\lambda_2$$

$$(ii) \sum X_{iu}^4 = 32 + 4a_1^4 + 4a_2^4 = cN\lambda_4$$

$$C. \sum X_{iu}^2 X_{ju}^2 = 32 = N\lambda_4 \quad (4.3)$$

From B (ii) and C of equation (4.1), we have  $c = \frac{32 + 4a_1^4 + 4a_2^4}{32}$

Substituting  $\lambda_2$ ,  $\lambda_4$  and c in equation (2.8) and on simplification, we get the following biquadratic equation

$$\begin{aligned} & 656(a_1^8 + a_2^8) + 1312a_1^4 a_2^4 - 384(a_1^6 a_2^2 + a_1^2 a_2^6) - 3072(a_1^6 + a_1^4 a_2^2 + a_1^2 a_2^4 + a_2^6) + 6400(a_1^4 + a_2^4) + 10240a_1^2 a_2^2 \\ & + 81920(a_1^2 + a_2^2) - 757760 = 0 \end{aligned}$$

Substitute  $a_1 = 1$  in the above equation and on simplification, we get

$$656a_2^8 - 3456a_2^6 + 4640a_2^4 + 88704a_2^2 - 671856 = 0 \quad (4.4)$$

Equation (4.4) has only one positive real root  $a_2^2 = 5.5686 \Rightarrow a_2 = 2.3598$ .

From the examples of (3.1) and (4.1), it can be noted that the value of  $a_2 = 2.3598$  in SOSRD of second type using CCD which is approximately nearer to the value of  $a_2 = 2.3596$  in SORD of second type using CCD by taking  $n_a = 2$ .

Note: For  $v=6$  factors in SOSRD of second type using CCD the design points are  $N=57$  and the value of  $a_2 = 2.9593$  with  $n_a = 1$ . (cf. Ravikumar and Victorbabu (2022)).

## 5. CONCLUSION

In this paper, second order slope rotatable designs (SOSRD) of second type using central composite designs (CCD) with  $2 \leq n_a \leq 4$  are suggested for  $2 \leq v \leq 17$ . It is observed that the value of level  $a_2$  for the axial points in CCD required for slope rotatability of second type is appreciably larger than the value required for second order rotatable designs (SORD) of second type using CCD. And also noted that if we replicate axial points ( $n_a$ ) in SOSRD of second type using CCD then the value of  $a_2$  is approximately nearer to SORD value  $a_2$  of second type using CCD.

The table gives the appropriate SRCCD values of the parameters  $a_2$  with  $a_1 = 1$  for designs using second type of CCD with  $2 \leq n_a \leq 4$  for  $2 \leq v \leq 17$  given in the Appendix.

## APPENDIX

Values of  $a_2$  for SOSRD of second type using CCD for  $2 \leq v \leq 17$  with  $2 \leq n_a \leq 4$

$v=2, p=0, a_1=1, a_2^*=1.3161$

| N  | $n_0$ | $n_a$ | $a_2$  |
|----|-------|-------|--------|
| 21 | 1     | 2     | 1.7347 |
| 25 | 5     | 2     | 1.5803 |
| 30 | 10    | 2     | 1.4738 |
| 35 | 15    | 2     | 1.4134 |
| 40 | 20    | 2     | 1.3755 |
| 45 | 25    | 2     | 1.3499 |
| 50 | 30    | 2     | 1.3315 |
| 55 | 35    | 2     | 1.3178 |
| 56 | 36    | 2     | 1.3154 |
| 29 | 1     | 3     | 1.6205 |
| 33 | 5     | 3     | 1.4818 |
| 38 | 10    | 3     | 1.3700 |
| 41 | 13    | 3     | 1.3238 |
| 42 | 14    | 3     | 1.3108 |
| 37 | 1     | 4     | 1.5473 |
| 41 | 5     | 4     | 1.4213 |
| 45 | 9     | 4     | 1.3276 |
| 46 | 10    | 4     | 1.3081 |

$v=3, p=0, a_1=1, a_2^*=1.6266$

| N  | $n_0$ | $n_a$ | $a_2$  |
|----|-------|-------|--------|
| 33 | 1     | 2     | 1.9110 |
| 37 | 5     | 2     | 1.8060 |
| 42 | 10    | 2     | 1.7321 |
| 47 | 15    | 2     | 1.6894 |
| 52 | 20    | 2     | 1.6624 |
| 57 | 25    | 2     | 1.6439 |
| 62 | 30    | 2     | 1.6304 |
| 63 | 31    | 2     | 1.6282 |
| 64 | 32    | 2     | 1.6260 |
| 45 | 1     | 3     | 1.7469 |
| 49 | 5     | 3     | 1.6554 |
| 50 | 6     | 3     | 1.6380 |
| 51 | 7     | 3     | 1.6222 |
| 57 | 1     | 4     | 1.6397 |
| 58 | 2     | 4     | 1.6165 |

$v=4, p=0, a_1=1, a_2^*=1.9680$

| N  | $n_0$ | $n_a$ | $a_2$  |
|----|-------|-------|--------|
| 49 | 1     | 2     | 2.1666 |
| 53 | 5     | 2     | 2.0991 |
| 58 | 10    | 2     | 2.0482 |
| 63 | 15    | 2     | 2.0167 |
| 68 | 20    | 2     | 1.9956 |
| 73 | 25    | 2     | 1.9806 |
| 78 | 30    | 2     | 1.9694 |

$v=5, p=1, a_1=1, a_2^*=1.9680$

| N  | $n_0$ | $n_a$ | $a_2$  |
|----|-------|-------|--------|
| 57 | 1     | 2     | 2.0781 |
| 61 | 5     | 2     | 2.0368 |
| 66 | 10    | 2     | 2.0061 |
| 71 | 15    | 2     | 1.9868 |
| 76 | 20    | 2     | 1.9736 |
| 78 | 22    | 2     | 1.9694 |
| 79 | 23    | 2     | 1.9675 |

|    |    |   |        |
|----|----|---|--------|
| 79 | 31 | 2 | 1.9675 |
| 65 | 1  | 3 | 1.9500 |
| 81 | 1  | 4 | 1.8046 |

$v=6, p=1, a_1=1, a_2^*=2.3596$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 81  | 1     | 2     | 2.4579 |
| 85  | 5     | 2     | 2.4269 |
| 90  | 10    | 2     | 2.4012 |
| 95  | 15    | 2     | 2.3837 |
| 100 | 20    | 2     | 2.3710 |
| 105 | 25    | 2     | 2.3614 |
| 106 | 26    | 2     | 2.3598 |
| 107 | 27    | 2     | 2.3582 |
| 105 | 1     | 3     | 2.1923 |
| 129 | 1     | 4     | 2.0191 |

|    |    |   |        |
|----|----|---|--------|
| 86 | 30 | 2 | 1.9567 |
| 77 | 1  | 3 | 1.8507 |
| 97 | 1  | 4 | 1.7009 |

$v=7, p=1, a_1=1, a_2^*=2.8173$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 121 | 1     | 2     | 2.9250 |
| 125 | 5     | 2     | 2.8989 |
| 130 | 10    | 2     | 2.8750 |
| 135 | 15    | 2     | 2.8574 |
| 140 | 20    | 2     | 2.8439 |
| 145 | 25    | 2     | 2.8334 |
| 150 | 30    | 2     | 2.8249 |
| 155 | 35    | 2     | 2.8179 |
| 156 | 36    | 2     | 2.8166 |
| 149 | 1     | 3     | 2.6158 |
| 177 | 1     | 4     | 2.4127 |

$v=8, p=2, a_1=1, a_2^*=2.8173$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 129 | 1     | 2     | 2.8858 |
| 133 | 5     | 2     | 2.8674 |
| 138 | 10    | 2     | 2.8505 |
| 143 | 15    | 2     | 2.8380 |
| 148 | 20    | 2     | 2.8283 |
| 153 | 25    | 2     | 2.8205 |
| 155 | 27    | 2     | 2.8179 |
| 156 | 28    | 2     | 2.8166 |
| 161 | 1     | 3     | 2.5789 |
| 193 | 1     | 4     | 2.3811 |

$v=9, p=2, a_1=1, a_2^*=3.3570$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 201 | 1     | 2     | 3.4518 |
| 205 | 5     | 2     | 3.4342 |
| 210 | 10    | 2     | 3.4166 |
| 215 | 15    | 2     | 3.4026 |
| 220 | 20    | 2     | 3.3913 |
| 230 | 30    | 2     | 3.3742 |
| 235 | 35    | 2     | 3.3676 |
| 240 | 40    | 2     | 3.3619 |
| 245 | 45    | 2     | 3.3570 |
| 237 | 1     | 3     | 3.0929 |
| 273 | 1     | 4     | 2.8574 |

$v=10, p=3, a_1=1, a_2^*=3.3570$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 209 | 1     | 2     | 3.4253 |
| 213 | 5     | 2     | 3.4114 |
| 218 | 10    | 2     | 3.3976 |
| 223 | 15    | 2     | 3.3866 |
| 228 | 20    | 2     | 3.3777 |
| 233 | 25    | 2     | 3.3703 |
| 238 | 30    | 2     | 3.3641 |
| 243 | 35    | 2     | 3.3589 |
| 245 | 37    | 2     | 2.3570 |
| 249 | 1     | 3     | 3.0673 |
| 289 | 1     | 4     | 2.8358 |

$v=11, p=4, a_1=1, a_2^*=3.3570$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 217 | 1     | 2     | 3.4026 |
| 221 | 5     | 2     | 3.3922 |
| 226 | 10    | 2     | 3.3818 |
| 231 | 15    | 2     | 3.3735 |
| 236 | 20    | 2     | 3.3666 |
| 241 | 25    | 2     | 3.3609 |
| 245 | 29    | 2     | 3.3570 |
| 261 | 1     | 3     | 3.0487 |
| 305 | 1     | 4     | 2.8213 |

$v=12, p=4, a_1=1, a_2^*=3.9961$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 353 | 1     | 2     | 4.0763 |
| 357 | 5     | 2     | 4.0650 |
| 362 | 10    | 2     | 4.0530 |
| 367 | 15    | 2     | 4.0430 |
| 372 | 20    | 2     | 4.0344 |
| 377 | 25    | 2     | 4.0270 |
| 382 | 30    | 2     | 4.0149 |

$v=13, p=5, a_1=1, a_2^*=3.9961$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 361 | 1     | 2     | 4.0592 |
| 365 | 5     | 2     | 4.0497 |
| 370 | 10    | 2     | 4.0396 |
| 375 | 15    | 2     | 4.0311 |
| 380 | 20    | 2     | 4.0238 |
| 385 | 25    | 2     | 4.0175 |
| 390 | 30    | 2     | 4.0121 |

|     |    |   |        |
|-----|----|---|--------|
| 387 | 35 | 2 | 4.0205 |
| 392 | 40 | 2 | 4.0099 |
| 397 | 45 | 2 | 4.0055 |
| 402 | 50 | 2 | 4.0015 |
| 407 | 55 | 2 | 3.9979 |
| 412 | 60 | 2 | 3.9947 |
| 401 | 1  | 3 | 3.6589 |
| 449 | 1  | 4 | 3.3857 |

|     |    |   |        |
|-----|----|---|--------|
| 395 | 35 | 2 | 4.0073 |
| 400 | 40 | 2 | 4.0031 |
| 405 | 45 | 2 | 3.9993 |
| 409 | 49 | 2 | 3.9966 |
| 410 | 50 | 2 | 3.9959 |
| 413 | 1  | 3 | 3.6422 |
| 465 | 1  | 4 | 3.3718 |

$v=14, p=6, a_1=1, a_2^*=3.9961$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 369 | 1     | 2     | 4.0437 |
| 373 | 5     | 2     | 4.0359 |
| 378 | 10    | 2     | 4.0276 |
| 383 | 15    | 2     | 4.0205 |
| 388 | 20    | 2     | 4.0145 |
| 393 | 25    | 2     | 4.0093 |
| 398 | 30    | 2     | 4.0048 |
| 403 | 35    | 2     | 4.0008 |
| 408 | 40    | 2     | 3.9972 |
| 409 | 41    | 2     | 3.9966 |
| 410 | 42    | 2     | 3.9959 |
| 425 | 1     | 3     | 3.6289 |
| 481 | 1     | 4     | 3.3615 |

$v=15, p=7, a_1=1, a_2^*=3.9961$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 377 | 1     | 2     | 4.0303 |
| 381 | 5     | 2     | 4.0240 |
| 386 | 10    | 2     | 4.0173 |
| 391 | 15    | 2     | 4.0116 |
| 396 | 20    | 2     | 4.0066 |
| 401 | 25    | 2     | 4.0024 |
| 406 | 30    | 2     | 3.9986 |
| 409 | 33    | 2     | 3.9966 |
| 410 | 34    | 2     | 3.9959 |
| 437 | 1     | 3     | 3.6186 |
| 497 | 1     | 4     | 3.3540 |

$v=16, p=8, a_1=1, a_2^*=3.9961$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 385 | 1     | 2     | 4.0191 |
| 389 | 5     | 2     | 4.0141 |
| 394 | 10    | 2     | 4.0087 |
| 399 | 15    | 2     | 4.0041 |
| 404 | 20    | 2     | 4.0001 |
| 409 | 25    | 2     | 3.9966 |
| 410 | 26    | 2     | 3.9959 |
| 449 | 1     | 3     | 3.6107 |
| 513 | 1     | 4     | 3.3485 |

$v=17, p=9, a_1=1, a_2^*=3.9961$

| N   | $n_0$ | $n_a$ | $a_2$  |
|-----|-------|-------|--------|
| 393 | 1     | 2     | 4.0099 |
| 397 | 5     | 2     | 4.0059 |
| 402 | 10    | 2     | 4.0016 |
| 407 | 15    | 2     | 3.9979 |
| 409 | 17    | 2     | 3.9966 |
| 410 | 18    | 2     | 3.9959 |
| 461 | 1     | 3     | 3.6048 |
| 529 | 1     | 4     | 3.3445 |

Note:  $a_2^*$  indicates the value of  $a_2$  in SORD of second type using CCD (cf. Kim (2002))

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