The equity-based modeling for two-player gambler's ruin problem

Abstract

In the context of the classical two-player gambler's ruin problem, the winning probabilities and initial stakes are pre-decided. If a player (who is in financial crisis) starts with less amount than his/her opponent in the symmetric game, has more chances to be ruined. Besides, a player (based on previous record data) with more winning probability than his/her competitor, has fewer chances to be ruined. We observe that most of the time, usually a weaker player is not fully willing to make a contest with a strong player. To give a fair chance to fight back for a weaker player and to develop the audience's interest, equity-based modeling is required. In this research, we propose some new equity-based models for the game of two players. In this way, we advocate the weaker player (with less winning probability or less amount to start the game) is motivated to participate in the contest because of a fair chance to make a comeback. The working methodology of newly proposed schemes is executed by deriving general expressions of the ruin probabilities for mathematical evaluation along with observing the ruin times, and then findings are compared with the results of a classic two-player game. Hence, the prime objectives related to the study are achieved by taking diverse parametric settings in the favor of equity-based modeling.

Keywords: Equity; gambler's problem; ruin probability; ruin time; comparison.

1 Introduction

The pioneering work related to gambler's problem is coined with a well-known mathematician Pascal; see for example Hussain et al. (2022). For the significance about this problem with applications in various disciplines, someone can consult Jayasekera (2018) for business, Lardizabal and Souza (2016) for quantum mechanics, Lorek et al. (2017) for anomaly detection, Cheng et al. (2007) for material properties and Petteruti (2017) for biological phenomena, etc.

The simplest form of this classic game is a contest between two players, where after each round, one of the players either win a pre-decided stake of the capital with winning probability p or lose with losing probability $1 - p = q$. As an economist or a statistician, our main interest lies to estimates the ruin probability and the expected ruin time of the game. Hussain et al. (2020) has provided the ruin probability of absorption to zero and we denoted here by "μ_k" and the average time to ruin with denoted by "D_k" are as:

$$\mu_k = \begin{cases} \frac{a-k}{a}; & q = p \\ \\ \frac{(r)^k-(r)^a}{1-(r)^a}; & q \neq p, \end{cases} \tag{1}$$

and,

$$D_k = \begin{cases} k(a-k); & q = p \\ \\ \frac{1}{q-p}\Big(\frac{a((r)^k-1)}{1-(r)^a} + k\Big); & q \neq p, \end{cases} \tag{2}$$

where k is the initial stake by gambler, a is denoted the total stakes involved in the game, and $r = \frac{q}{p}$.

Recently, several new trends have been observed about this problem, such as, Kmet and Petkovsek (2002) proposed a game plan with the involvement of unlimited stakes. Lefebvre (2008) and El-Shehawey

(2009) introduced some procedures to measure the probabilities by taking current stakes of the competitors. Lengyel (2009) extended the game duration only by involving the ties in this game. One more variant about this problem in the literature is, a successive trials based decision for winning or losing the amount by Hussain et al. (2020) and Hussain and Cheema (2021). They showed that the average duration is not only extended but also with reducing the probability of ruin as well. Also, they claimed that their strategy is in favor of both competitors. The most recent advancement in the literature, Hussain et al. (2022) advocated the use of simultaneous operation of multiple devices to conclude this game.

In this research article, we propose some new amendments with equity based transmission in the classical two-player game. With the arguments by these new game plans, we want to provide a fair chance of come back for a weaker competitor. In Section 2, we provide the mathematical developments for the expected ruin probability and duration for our proposed Model-I, for Model-II in Section 3 and for Model-III in Section 4. To measure the performance evaluations through numerical study while considering several parametric setting in Section 5. Finally in Section 6, we have provided some concluding remarks about this research work.

2 Model-I: when gambler has more initial stakes

In this section, we are provided the explicit expressions for the ruin probability and expected duration of the game under the proposed Model-I, when the gambler has more initial stakes than the opponent in the symmetric case. Thus, after each trial of the game, the gambler can win a dollar with probability p. But on the other hand, the opponent can win twice amount after two trials with probability q^2. Besides that, one may appreciate that the involvement of ties in this new game plan at qp. In the next subsections, we derive the required expressions and initiate our calculations for the most practical and simple case, i.e. decision after one or two trials depending on the p and q, respectively and presented in the Figure 1.

2.1 The ruin probability

Let, μ_k be the ruin probability of the gambler, who starting the game with k dollars. The theorem of total probability can be written in the following difference equation (based on Figure 1) as:

$$\mu_k = \mu_{k+1}(p) + \mu_k(qp) + \mu_{k-2}(q^2).$$

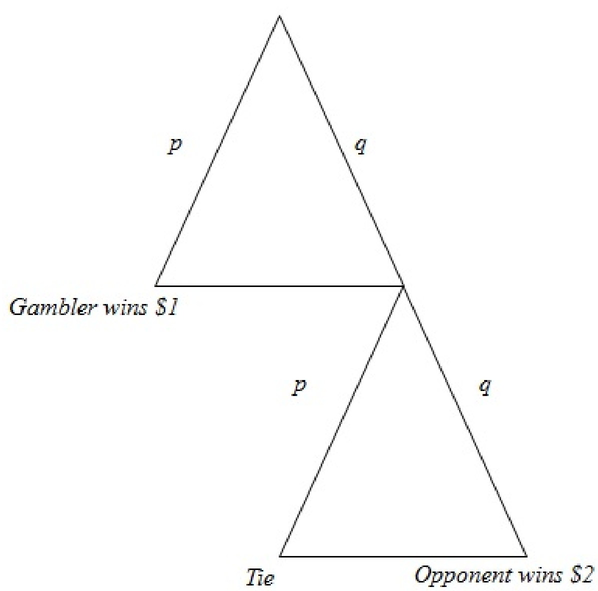

Figure 1: Proposed game plan for Model-I.

The most simple form of the above difference equation after implying $q = p = \frac{1}{2}$ as under:

$$2\mu_{k+1} - 3\mu_k + \mu_{k-2} = 0. \tag{3}$$

This equation has three roots as: $\eta_1 = \eta_2 = 1$ and $\eta_3 = \frac{-1}{2}$, so the general solution of equation (3) can be written as:

$$\mu_k = A_1 + A_2 k + A_3 (\frac{-1}{2})^k,$$

where A_1, A_2 and A_3 are constants and can be determined using the boundary conditions, $\mu_0 = \mu_1 = 1$ and $\mu_a = 0$, in the following forms:

$$A_1 = 1 + \frac{1}{\frac{3a}{2} - 1 + (\frac{-1}{2})^a},$$

$$A_2 = \frac{\frac{-3}{2}}{\frac{3a}{2} - 1 + (\frac{-1}{2})^a},$$

and

$$A_3 = \frac{-1}{\frac{3a}{2} - 1 + (\frac{-1}{2})^a}.$$

Thus the general solution to the equation (3) is:

$$\mu_k = A_1 + A_2 k + A_3 (\frac{-1}{2})^k. \tag{4}$$

The ruin probability for the proposed game scenario can also be extended to make a decision after three, four or even more trials but this would lead difficulty in computations.

2.2 The ruin time

The general expression for the calculation of expected ruin time, say D_k can be written as:

$$D_k = \sum_{n=0}^{\infty} n \; P(n|k),$$

where n is required number of steps to complete the game and k is the initial amount of the gambler. By employing the proposed Model-I scheme, i.e. in symmetric game, the gambler has more initial stakes. Under the law of total probability, the $P(n|k)$ can be written as:

$$P(n|k) = P(n-1|k+1)(p) + P(n-2|k)(qp) + P(n-2|k-2)(q^2).$$

After using expectation operation, the above equation reduces to:

$$D_k = p \sum_{n=1}^{\infty} nP(n-1|k+1) + qp \sum_{n=2}^{\infty} nP(n-2|k) + q^2 \sum_{n=2}^{\infty} nP(n-2|k-2).$$

For one step back, we imply $n - 1 = m$ and the above equation can be written as:

$$D_k = p \sum_{m=0}^{\infty} (m+1)P(m|k+1) + qp \sum_{m=1}^{\infty} (m+1)P(m-1|k) + q^2 \sum_{m=1}^{\infty} (m+1)P(m-1|k-2).$$

$$D_k = p \sum_{m=0}^{\infty} mP(m|k+1) + p \sum_{m=0}^{\infty} P(m|k+1) + qp \sum_{m=1}^{\infty} (m+1)P(m-1|k) + q^2 \sum_{m=1}^{\infty} (m+1)P(m-1|k-2).$$

$$D_k = pD_{k+1} + p + qp\sum_{m=1}^{\infty}(m+1)P(m-1|k) + q^2\sum_{m=1}^{\infty}(m+1)P(m-1|k-2).$$

where, $\sum_{m=0}^{\infty} P(m|k+1) = 1$. Again, for one step back, we imply $m-1=r$ and the above equation can be written as:

$$D_k = pD_{k+1} + p + qp\sum_{r=0}^{\infty}(r+2)P(r|k) + q^2\sum_{r=0}^{\infty}(r+2)P(r|k-2).$$

$$D_k = pD_{k+1} + p + qpD_k + 2qp + q^2D_{k-2} + 2q^2.$$

$$pD_{k+1} + (qp-1)D_k + q^2D_{k-2} = -(1+q)$$

For the case of symmetric game, we imply $q = p = \frac{1}{2}$, so that the above equation can be written as:

$$2D_{k+1} - 3D_k + D_{k-2} = -6. \tag{5}$$

The equation (5) is an inhomogeneous difference equation, so it has two solutions, i.e. the complementary and the particular. First, we consider the characteristic equation for the complementary solution as:

$$2D_{k+1} - 3D_k + D_{k-2} = 0.$$

The above equation has three roots as: $\eta_1 = \eta_2 = 1$ and $\eta_3 = \frac{-1}{2}$, so the complementary solution of equation (5) can be written as:

$$D_k = B_1 + B_2k + B_3(\frac{-1}{2})^k.$$

The particular solution is remains $D_k = B_4k^2$, and determined B_4 in the following way:

$$2B_4(k+1)^2 - 3B_4k^2 + B_4(k-2)^2 = -6,$$

and $B_4 = -1$, and particular solution is: $D_k = -k^2$. So, the general solution is:

$$D_k = B_1 + B_2k + B_3(\frac{-1}{2})^k - k^2,$$

where B_1, B_2 and B_3 are constants and can be determined by using the three boundary conditions, such as: $D_0 = D_1 = D_a = 0$, and the constants are:

$$B_1 = \frac{-a(1-a)}{1 - (\frac{-1}{2})^a - \frac{3a}{2}},$$

$$B_2 = 1 + \frac{\frac{3a}{2}(1-a)}{1 - (\frac{-1}{2})^a - \frac{3a}{2}},$$

and

$$B_3 = \frac{a(1-a)}{1 - (\frac{-1}{2})^a - \frac{3a}{2}}.$$

So, the general solution will be in the following form:

$$D_k = B_1 + B_2k + B_3(\frac{-1}{2})^k - k^2. \tag{6}$$

3 Model-II: when gambler has less initial stakes

In this section, we are provided the explicit expressions for the ruin probability and expected duration of the game under the proposed Model-II, when the gambler has less initial stakes than his/her opponent in symmetric case. Thus, after each trial of the game, the opponent can win a dollar with probability q. But on the other hand, the gambler can win twice amount after two trials with probability p^2. Besides that, one may appreciate that the involvement of ties in this new game plan at pq. The complete scenario about Model-II is presented in the Figure 2.

3.1 The ruin probability

Let, μ_k be the ruin probability of the gambler, who starting the game with k dollars. The theorem of total probability can be written in the following difference equation (based on Figure 2) as:

$$\mu_k = \mu_{k-1}(q) + \mu_k(pq) + \mu_{k+2}(p^2).$$

The most simple form of the above difference equation after implying $p = q = \frac{1}{2}$ as under:

$$\mu_{k+2} - 3\mu_k + 2\mu_{k-1} = 0. \tag{7}$$

The equation (7) has three roots as: $\eta_1 = \eta_2 = 1$ and $\eta_3 = -2$, so the general solution can be written as:

$$\mu_k = C_1 + C_2 k + C_3(-2)^k,$$

where, C_1, C_2 and C_3 are constants and can be determined using the boundary conditions, $\mu_0 = 1$ and $\mu_a = \mu_{a-1} = 0$, in the following forms:

$$C_1 = \frac{(a-1)(-2)^a - a(-2)^{a-1}}{1 + (a-1)(-2)^a - a(-2)^{a-1}},$$

$$C_2 = \frac{(-2)^{a-1} - (-2)^a}{1 + (a-1)(-2)^a - a(-2)^{a-1}},$$

and

$$C_3 = \frac{1}{1 + (a-1)(-2)^a - a(-2)^{a-1}}.$$

Thus the general solution to the equation (7) is:

$$\mu_k = \frac{(-2)^a(a-k-1) - (-2)^{a-1}(a-k) + (-2)^k}{1 + (a-1)(-2)^a - a(-2)^{a-1}}. \tag{8}$$

The ruin probability for the proposed game scenario can also be extended to make a decision after three, four or even more trials but this would lead difficulty in computations.

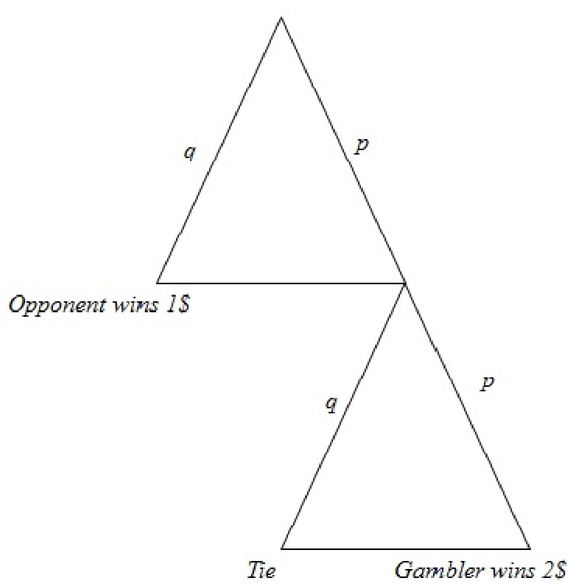

Figure 2: Proposed game plan for Model-II.

3.2 The ruin time

The general expression for the calculation of expected ruin time can be written as:

$$D_k = \sum_{n=0}^{\infty} n \; P(n|k),$$

where n and k are the required number of steps to complete the game and the initial amount of the gambler, respectively. By employing Model-II scheme under the law of total probability, the $P(n|k)$ can be written as:

$$P(n|k) = P(n-1|k-1)(q) + P(n-2|k)(pq) + P(n-2|k+2)(p^2).$$

After using expectation operation, the above equation reduces to:

$$D_k = q\sum_{n=1}^{\infty} nP(n-1|k-1) + pq\sum_{n=2}^{\infty} nP(n-2|k) + p^2\sum_{n=2}^{\infty} nP(n-2|k+2).$$

For one step back, we imply $n-1=m$ and the above equation can be written as:

$$D_k = q\sum_{m=0}^{\infty}(m+1)P(m|k-1) + pq\sum_{m=1}^{\infty}(m+1)P(m-1|k) + p^2\sum_{m=1}^{\infty}(m+1)P(m-1|k+2).$$

$$D_k = q\sum_{m=0}^{\infty} mP(m|k-1) + q\sum_{m=0}^{\infty} P(m|k-1) + pq\sum_{m=1}^{\infty}(m+1)P(m-1|k) + p^2\sum_{m=1}^{\infty}(m+1)P(m-1|k+2).$$

$$D_k = qD_{k-1} + q + pq\sum_{m=1}^{\infty}(m+1)P(m-1|k) + p^2\sum_{m=1}^{\infty}(m+1)P(m-1|k+2),$$

where $\sum_{m=0}^{\infty} P(m|k-1) = 1$. Again, for one step back, we imply $m-1=r$ and the above equation can be written as:

$$D_k = qD_{k-1} + q + pq\sum_{r=0}^{\infty}(r+2)P(r|k) + p^2\sum_{r=0}^{\infty}(r+2)P(r|k+2).$$

$$D_k = qD_{k-1} + q + pqD_k + 2pq + p^2D_{k+2} + 2p^2.$$

$$p^2D_{k+2} + (pq-1)D_k + qD_{k-1} = -(1+p). \tag{9}$$

For the symmetric case, we imply $p=q=\frac{1}{2}$, so that the equation (9) can be written as:

$$D_{k+2} - 3D_k + 2D_{k-1} = -6. \tag{10}$$

The equation (10) is an inhomogeneous difference equation, so it has two solutions, i.e. the complementary and the particular. First, we consider the characteristic equation for the complementary solution as:

$$D_{k+2} - 3D_k + 2D_{k-1} = 0.$$

The above equation has three roots as: $\eta_1 = \eta_2 = 1$ and $\eta_3 = -2$, so the complementary solution of equation (10) can be written as:

$$D_k = E_1 + E_2k + E_3(-2)^k.$$

The particular solution is remains $D_k = E_4k^2$, and determined E_4 in the following way:

$$E_4(k+2)^2 - 3E_4k^2 + 2E_4(k-1)^2 = -6,$$

and $E_4 = -1$, and particular solution is: $D_k = -k^2$. So, the general solution is:

$$D_k = E_1 + E_2k + E_3(-2)^k - k^2,$$

where E_1, E_2 and E_3 are constants and can be determined using the three boundary conditions, such as: $D_0 = D_a = D_{a-1} = 0$, and the constants are:

$$E_1 = \frac{-a(a-1)}{1 + (a-1)(-2)^a - a(-2)^{a-1}},$$

$$E_2 = \frac{1 - (-2)^a(1 + a^2) + a^2(-2)^{a-1}}{1 + (a-1)(-2)^a - a(-2)^{a-1}},$$

and

$$E_3 = \frac{a(a-1)}{1 + (a-1)(-2)^a - a(-2)^{a-1}}.$$

So, the general solution will be in the following form:

$$D_k = \frac{k(1-k) - a(a-1) + a(a-1)(-2)^k + k\big((-2)^a(1 + a^2) + a^2(-2)^{a-1}\big)}{1 + (a-1)(-2)^a - a(-2)^{a-1}}. \tag{11}$$

4 Model-III: significant difference between winning probabilities

In this section, we are given the explicit expressions for the ruin probability and expected duration of the game under the proposed scenario of Model-III. Thus, after each trial of the game, the opponent (specifically a new player or comparatively weaker than gambler) can win a dollar with probability q. But on the other hand, the gambler can only win the same amount after two trials with probability p^2. Besides that, one may appreciate that no involvement of ties in this new game plan. In the next subsections, we derive the required expressions and initiate our calculations for the most practical and simple case, i.e. decision after one or two trials depending on the outcomes of q and p, respectively. This game plan is graphically presented in Figure 3.

4.1 The ruin probability

Let, μ_k be the ruin probability of the gambler, who starting the game with k dollars. The theorem of total probability can be written in the following difference equation (based on Figure 3) as:

$$\mu_k = \mu_{k-1}(q) + \mu_{k-1}(pq) + \mu_{k+1}(p^2).$$

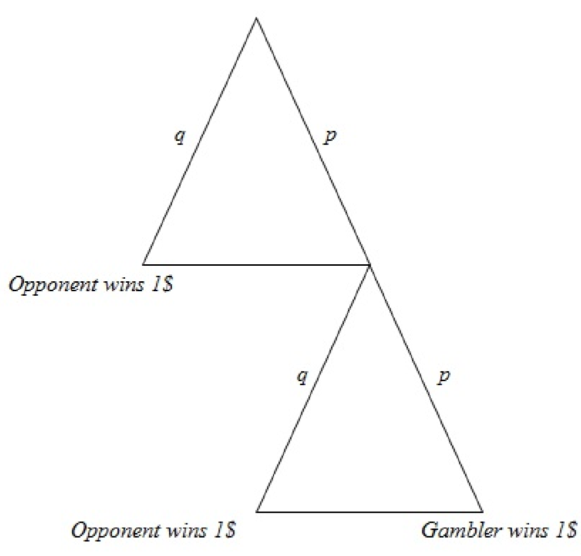

Figure 3: Proposed game plan for Model-III.

The most simple form of the above difference equation as under:

$$(p^2)\mu_{k+1} + \mu_k + (pq+q)\mu_{k-1} = 0. \tag{12}$$

The equation (12) has two different roots as:

$$\lambda_1 = \frac{1+\sqrt{1-4(p+1)qp^2}}{2p^2},$$

and

$$\lambda_2 = \frac{1-\sqrt{1-4(p+1)qp^2}}{2p^2}.$$

The general solution of equation (12) can be written as:

$$\mu_k = F_1\left(\frac{1+\sqrt{1-4(p+1)qp^2}}{2p^2}\right)^k + F_2\left(\frac{1-\sqrt{1-4(p+1)qp^2}}{2p^2}\right)^k,$$

where F_1 and F_2 are constants and can be determined by using the boundary conditions, i.e. $\mu_0 = 1$ and $\mu_a = 0$, in the following forms:

$$F_1 = -\frac{\left(1-\sqrt{1-4(p+1)qp^2}\right)^a}{\left(1+\sqrt{1-4(p+1)qp^2}\right)^a - \left(1-\sqrt{1-4(p+1)qp^2}\right)^a},$$

and

$$F_2 = \frac{\left(1+\sqrt{1-4(p+1)qp^2}\right)^a}{\left(1+\sqrt{1-4(p+1)qp^2}\right)^a - \left(1-\sqrt{1-4(p+1)qp^2}\right)^a}.$$

Thus the general solution to the equation (12) is:

$$\mu_k = \frac{-\left(1-\sqrt{1-4(p+1)qp^2}\right)^a\left(\frac{1+\sqrt{1-4(p+1)qp^2}}{2p^2}\right)^k + \left(1+\sqrt{1-4(p+1)qp^2}\right)^a\left(\frac{1-\sqrt{1-4(p+1)qp^2}}{2p^2}\right)^k}{\left(1+\sqrt{1-4(p+1)qp^2}\right)^a - \left(1-\sqrt{1-4(p+1)qp^2}\right)^a}. \tag{13}$$

The ruin probability for this proposed game scenario can also be extended to make a decision after three, four or even more trials but this would lead difficulty in computations.

4.2 The ruin time

The general expression for the calculation of expected ruin time can be written as:

$$D_k = \sum_{n=0}^{\infty} n\ P(n|k), \tag{14}$$

where, n is required number of steps to complete the game and k is the initial amount of the gambler. By employing Model-III scheme under the law of total probability, the $P(n|k)$ can be written as:

$$P(n|k) = P(n-1|k-1)(q) + P(n-2|k-1)(pq) + P(n-2|k+1)(p^2).$$

After using expectation operation, the above equation reduces to:

$$D_k = q\sum_{n=1}^{\infty} nP(n-1|k-1) + pq\sum_{n=2}^{\infty} nP(n-2|k-1) + p^2\sum_{n=2}^{\infty} nP(n-2|k+1).$$

For one step back, we imply $n-1=m$ and the above equation can be written as:

$$D_k = q\sum_{m=0}^{\infty}(m+1)P(m|k-1) + pq\sum_{m=1}^{\infty}(m+1)P(m-1|k-1) + p^2\sum_{m=1}^{\infty}(m+1)P(m-1|k+1).$$

$$D_k = q\sum_{m=0}^{\infty} mP(m|k-1) + q\sum_{m=0}^{\infty} P(m|k-1) + pq\sum_{m=1}^{\infty}(m+1)P(m-1|k-1) + p^2\sum_{m=1}^{\infty}(m+1)P(m-1|k+1).$$

$$D_k = qD_{k-1} + q + pq\sum_{m=1}^{\infty}(m+1)P(m-1|k-1) + p^2\sum_{m=1}^{\infty}(m+1)P(m-1|k+1),$$

where $\sum_{m=0}^{\infty} P(m|k-1) = 1$. Again, for one step back, we imply $m-1=r$ and the above equation can be written as:

$$D_k = qD_{k-1} + q + pq\sum_{r=0}^{\infty}(r+2)P(r|k-1) + p^2\sum_{r=0}^{\infty}(r+2)P(r|k+1).$$

$$D_k = qD_{k-1} + q + pqD_{k-1} + 2pq + p^2D_{k+1} + 2p^2.$$

$$p^2D_{k+1} - D_k + (pq+q)D_{k-1} = -(1+p). \tag{15}$$

The equation (15) is an inhomogeneous difference equation, so it has two solutions, i.e. the complementary and the particular. First, we consider the characteristic equation for the complementary solution as:

$$p^2D_{k+1} - D_k + (pq+q)D_{k-1} = 0. \tag{16}$$

The equation (16) gives two different roots as:

$$\theta_1 = \frac{1+\sqrt{1-4(p+1)qp^2}}{2p^2},$$

and

$$\theta_2 = \frac{1-\sqrt{1-4(p+1)qp^2}}{2p^2},$$

and the complementary solution is:

$$D_k = G_1(\theta_1)^k + G_2(\theta_2)^2.$$

The particular solution is remains $D_k = G_3k$, and determined G_3 in the following way:

$$p^2G_3(k+1) - G_3(k) + (pq+q)G_3(k-1) = -(1+p).$$

$$G_3 = \frac{-(1+p)}{p^2-pq-q},$$

and the solution is: $D_k = \frac{-(1+p)k}{p^2-pq-q}$. The combined solution will be given as:

$$D_k = G_1(\theta_1)^k + G_2(\theta_2)^2 + \frac{-(1+p)k}{p^2-pq-q},$$

where G_1 and G_2 are constants and can be determined by using the two boundary conditions, such as: $D_0 = D_a = 0$, and the constants are:

$$G_1 = \frac{(1+p)a}{(\theta_1^a - \theta_2^a)(p^2 - pq - q)},$$

and

$$G_2 = \frac{-(1+p)a}{(\theta_1^a - \theta_2^a)(p^2 - pq - q)}.$$

The final form of the D_k is:

$$D_k = \frac{1+p}{p^2 - pq - q}\left(a\frac{\theta_1^k - \theta_2^k}{\theta_1^a - \theta_2^a} - k\right), \tag{17}$$

where θ_1 and θ_2 are defined above.

5 Numerical evaluation

In this section, we examined the evaluation of the proposed models, which are derived in previous sections. The study is conducted to determine the required goals by three proposed schemes based on equity modeling. The first goal is when gambler has more amount than his/her opponent in the symmetric game, then how can we increase the involvement of audience than a traditional game plan. This goal has been achieved through Model-I. The second goal is reciprocal of the first goal and has been achieved by introducing Model-II. Model-III is based on asymmetric game plan with a player has more probability (say 70% or more) to win a trial. In the classical game scenario, there is almost no chance for other player to win the game. By providing the Model-III, we have achieved our third goal, i.e. to make a possible comeback for that player, who played with less winning probability (say 30% or less).

For comparison purposes, we present Tables 1-3 with the values of the ruin probabilities and the ruin times under Model-I, Model-II, and Model-III, respectively. Based on demonstration purposes, we considered, $a = 10$, $a = 20$ and $a = 30$ to compare with the original results of two-player gambler's ruin problem. According to the results of Tables 1, 2, and 3 reveal that as we increase the amount of initial stake, the ruin probabilities gradually decrease along with their average ruin time. Moreover, ruin probabilities and average ruin time are also comparatively increased when we included more total stake amount in the game under Model-I. Similarly, the increment in the ruin probabilities and ruin time is also repeated under Model-II and Model-III as well. We observed that all the proposed models are very useful to enhance the interest of the comparative weaker player in the game. These models are also useful to reinforce the audience's interest to watch the game till end. The comparisons are also presented through graphics between proposed models and traditional ones in Figures 4 to 6, for Model-I, Model-II, and Model-III, respectively.

Table 1: Comparisons of the ruin probability and ruin time between the classical ruin game and the proposed Model-I. Model-I results are in brackets.

	μ_k			D_k		
k	a=10	a=20	a=30	a=10	a=20	a=30
6	0.4(**0.43**)			24(**21.53**)		
7	0.3(**0.32**)			21(**19.02**)		
8	0.2(**0.21**)			16(**14.73**)		
9	0.1(**0.11**)			9(**8.33**)		
11		0.45(**0.47**)			99(**93.10**)	
12		0.40(**0.41**)			96(**90.76**)	
13		0.35(**0.36**)			91(**86.41**)	
14		0.30(**0.31**)			84(**80.07**)	
15		0.25(**0.26**)			75(**71.72**)	
16		0.20(**0.21**)	0.47(**0.48**)		64(**61.38**)	224(**214.77**)
17		0.15(**0.16**)	0.43(**0.44**)		51(**49.03**)	221(**212.43**)
18		0.10(**0.103**)	0.40(**0.41**)		36(**34.69**)	216(**208.09**)
19		0.05(**0.052**)	0.37(**0.38**)		19(**18.34**)	209(**201.75**)
20			0.33(**0.34**)			200(**193.41**)
21			0.30(**0.31**)			189(**183.07**)
22			0.267(**0.273**)			176(**170.73**)
23			0.233(**0.239**)			161(**156.39**)
24			0.200(**0.205**)			144(**140.05**)
25			0.167(**0.170**)			125(**121.70**)
26			0.133(**0.136**)			104(**101.36**)
27			0.100(**0.102**)			81(**79.02**)
28			0.067(**0.068**)			56(**54.68**)
29			0.033(**0.034**)			29(**28.34**)

Table 2: Comparisons of the ruin probability and ruin time between the traditional ruin game and the proposed Model-II. Model-II results are in brackets.

	μ_k			D_k		
k	a=10	a=20	a=30	a=10	a=20	a=30
1	0.90(**0.89**)	0.95(**0.95**)	0.967(**0.966**)	9(**3.62**)	19(**6.93**)	29(**10.25**)
2	0.80(**0.79**)	0.90(**0.90**)	0.933(**0.932**)	16(**7.30**)	36(**13.86**)	56(**20.50**)
3	0.70(**0.68**)	0.85(**0.84**)	0.900(**0.898**)	21(**10.87**)	51(**20.79**)	81(**30.75**)
4	0.60(**0.57**)	0.80(**0.79**)	0.868(**0.864**)	24(**14.66**)	64(**27.72**)	104(**41.00**)
5	0.50(**0.46**)	0.75(**0.74**)	0.833(**0.830**)	25(**18.00**)	75(**34.65**)	125(**51.25**)
6		0.70(**0.69**)	0.800(**0.795**)		84(**41.59**)	144(**61.50**)
7		0.65(**0.64**)	0.767(**0.761**)		91(**48.52**)	161(**71.75**)
8		0.60(**0.59**)	0.733(**0.727**)		96(**55.45**)	176(**82.00**)
9		0.55(**0.53**)	0.700(**0.693**)		99(**62.37**)	189(**92.25**)
10		0.50(**0.48**)	0.667(**0.659**)		100(**69.32**)	200(**102.50**)
11			0.633(**0.625**)			209(**112.75**)
12			0.600(**0.591**)			216(**123.00**)
13			0.567(**0.557**)			221(**133.25**)
14			0.533(**0.522**)			224(**143.50**)
15			0.500(**0.489**)			225(**153.75**)

Table 3: Comparisons of the ruin probability and ruin time between the traditional ruin game and the proposed Model-III. Model-III results are in brackets.

p	μ_k			D_k		
	a=10	a=20	a=30	a=10	a=20	a=30
0.71	0.011(**0.480**)	0.000129(**0.459091**)	0.000001(**0.438807**)	11.64(**42.73**)	23.80(**170.62**)	35.71(**382.83**)
0.72	0.009(**0.409**)	0.000079(**0.323807**)	0.000001(**0.248898**)	11.16(**42.54**)	22.72(**164.70**)	34.09(**352.09**)
0.73	0.007(**0.341**)	0.000048(**0.211167**)	0.000000(**0.121654**)	10.72(**41.81**)	21.74(**151.88**)	32.61(**298.42**)
0.74	0.005(**0.278**)	0.000029(**0.129005**)	0.000000(**0.053928**)	10.31(**40.59**)	20.83(**135.62**)	31.25(**244.59**)
0.75	0.004(**0.221**)	0.000017(**0.074942**)	0.000000(**0.022539**)	9.92(**38.98**)	20.00(**119.02**)	30.00(**200.53**)
0.76	0.003(**0.173**)	0.000010(**0.041915**)	0.000000(**0.009068**)	9.56(**37.08**)	19.23(**103.90**)	28.85(**167.02**)
0.77	0.0023(**0.132**)	0.000006(**0.022761**)	0.000000(**0.003542**)	9.22(**35.02**)	18.52(**90.93**)	27.78(**141.88**)
0.78	0.0017(**0.0994**)	0.000003(**0.012058**)	0.000000(**0.001347**)	8.90(**32.88**)	17.86(**80.12**)	26.79(**122.82**)
0.79	0.0013(**0.0734**)	0.000002(**0.006244**)	0.000000(**0.000498**)	8.60(**30.76**)	17.24(**71.22**)	25.86(**108.07**)
0.80	0.0009(**0.0533**)	0.000001(**0.003161**)	0.000000(**0.000179**)	8.32(**28.72**)	16.67(**63.88**)	25.00(**96.39**)
0.81	0.0007(**0.0381**)	0.000001(**0.001563**)	0.000000(**0.000062**)	8.05(**26.78**)	16.13(**57.79**)	24.19(**86.95**)
0.82	0.0005(**0.0267**)	0.000000(**0.000753**)	0.000000(**0.000021**)	7.80(**24.98**)	15.62(**52.70**)	23.44(**79.17**)
0.83	0.0004(**0.0184**)	0.000000(**0.000353**)	0.000000(**0.000007**)	7.57(**23.33**)	15.15(**48.40**)	22.73(**72.66**)
0.84	0.0003(**0.0125**)	0.000000(**0.000160**)	0.000000(**0.000002**)	7.35(**21.81**)	14.71(**44.73**)	22.06(**67.12**)
0.85	0.0002(**0.0083**)	0.000000(**0.000070**)	0.000000(**0.000001**)	7.14(**20.44**)	14.29(**41.57**)	21.43(**62.36**)
0.86	0.0001(**0.0054**)	0.000000(**0.000029**)	0.000000(**0.000000**)	6.94(**19.20**)	13.89(**38.81**)	20.83(**58.22**)
0.87	0.0001(**0.0034**)	0.000000(**0.000012**)	0.000000(**0.000000**)	6.76(**18.07**)	13.51(**36.39**)	20.27(**54.59**)
0.88	0.0000(**0.0021**)	0.000000(**0.000004**)	0.000000(**0.000000**)	6.58(**17.06**)	13.16(**34.26**)	19.74(**51.38**)
0.89	0.0000(**0.0012**)	0.000000(**0.000002**)	0.000000(**0.000000**)	6.41(**16.14**)	12.82(**32.35**)	19.23(**48.53**)
0.90	0.0000(**0.0007**)	0.000000(**0.000001**)	0.000000(**0.000000**)	6.25(**15.30**)	12.50(**30.65**)	18.75(**45.97**)

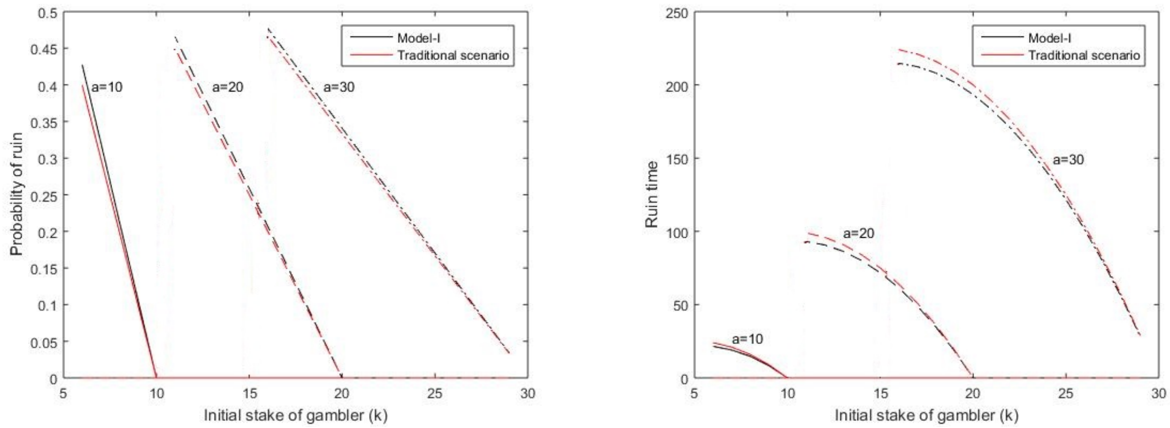

Figure 4: Comparison between Model-I and tradational game: **(a)** the ruin probability, **(b)** the ruin time

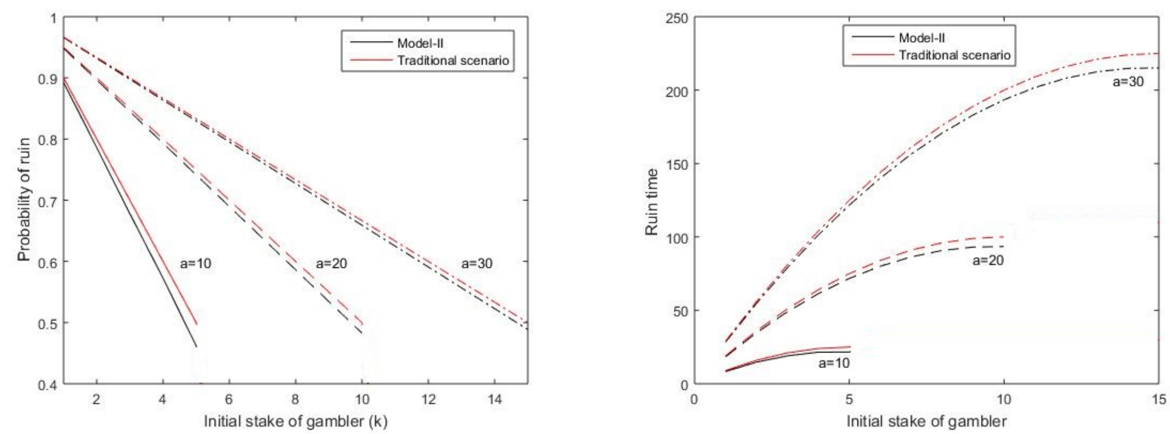

Figure 5: Comparison between Model-II and tradational game: **(a)** the ruin probability, **(b)** the ruin time

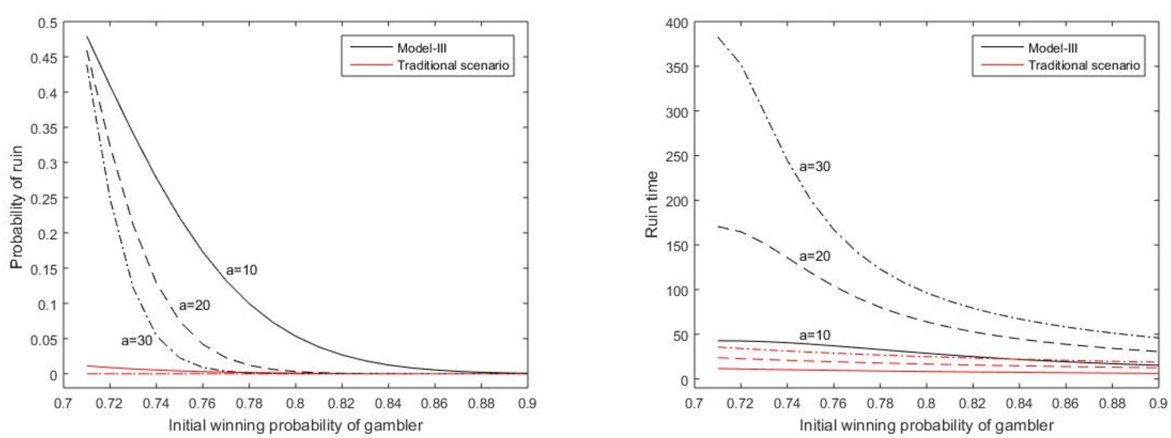

Figure 6: Comparison between Model-III and tradational game: **(a)** the ruin probability, **(b)** the ruin time

6 Conclusions

In this research, we present the classic two-player ruin's problem with several variants based on equity modeling. To the best of our knowledge, these types of game plans have never been discussed in the literature. Overall, the proposed equity-based models provide a significant contribution to those opponents who have limited resources for the competition. In fact, these models revitalize a hope of winning for comparatively weaker opponents. The rationale of the propositions thus providing a chance to make a possible comeback for the player, who is playing with a significantly less amount or with less probability to win a trial. The legitimacy of the devised approaches is mathematically established and rigorously verified with numerical results.

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