Internal Set Theory IST#based on hyper infinitary logic with Restricted Modus Ponens Rule.Nonconservative extension of the model theoretical NSA.

Abstract: The incompleteness of set theory ZFC leads one to look for natural nonconservative extensions of ZFC in which one can prove statements independent of ZFC which appear to be “true”. One approach has been to add large cardinal axioms.
Or, one can investigate second-order expansions like Kelley-Morse class theory, KM or
Tarski-Grothendieck set theory TG or It is a nonconservative extension of ZFC and is obtained from other axiomatic set theories by the inclusion of Tarski's axiom which implies the existence of inaccessible cardinals. See also related set theory with a filter quantifier $ZF\_aa\_$. In this paper we look at a set theory $\mathbf{NC}_{\#}^{\#}$,based on bivalent gyper infinitary logic with restricted Modus Ponens Rule In this paper we deal with set theory $\mathbf{NC}_{\#}^{\#}$ based on bivalent gyper infinitary logic with Restricted Modus Ponens Rule.Nonconservative extensions of the canonical internal set theories IST and HST are proposed.
Keywords:Set theory ZFC;Nonconservative extension of ZFC;Internal set theory IST;External set theory HST; A.Robinson model theoretical NSA; Bivalent gyper infinitary logic;Modus ponens rule; Logic with restricted modus ponens rule;
2010 Mathematics Subject Classification:

1.Introduction

The incompleteness of set theory ZFC leads one to look for nonconservative natural extensions of ZFC in which one can prove statements independent of ZFC which appear to be “true”.
One approach has been to add large cardinal axioms. Or, one can investigate second-order expansions like Kelley-Morse class theory, KM [1] or Tarski-Grothendieck set theory TG [2]. It is a non-conservative extension of ZFC and is obtained from other

inaccessible cardinals. See also set theory with a filter quantifier $ZF\_aa\_$ [3],related to set theory with satisfaction predicate.However nonconservative extensions of ZFC mentioned above related only to pure set theoretical statements.
In this paper we look for nonconservative extensions of ZFC in which one can prove statements related to number theory and analysis.
In this paper we deal with set theory $\mathbf{NC}_{\#}^{\#}$ based on hyper infinitary logic ${}^{2}L_{\#}^{\#}$ with restricted modus ponens rule [4]-[7].Non trivial applications of the set theory $\mathbf{NC}_{\#}^{\#}$ to transcendental number theory and functional analysis has been recently obtained in my papers [7]-[10].However all results obtained in [7]-[10] based on a small part of the set theory $\mathbf{NC}_{\#}^{\#}$ and in fact are obtained using an nonconservative extension of

the canonical internal set theory *IST* [11]-[13].
The main goal of this paper is to present an nonconservative extension $IST_{\#}$ of the canonical internal set theory *IST*.Nonconservative extension of the model theoretical nonstandard analysis also is considered.

2.Set theory **NC**_#

.
Set theory **NC**_#
is formulated as a system of axioms based on bivalent hyper
infinitary logic 2*L*_#
with restricted modus ponens rule [8],see Appendix A. The language
of set theory **NC**_#
is a first-order hyper infinitary language *L*_#
with equality _, which
includes a binary symbol _. We write *x* _ *y* for _ _*x* _ *y*_ and *x* _ *y* for __*x* _ *y*_.
Individual variables *x*, *y*, *z*, . . . of *L*_#
will be understood as ranging over classical sets. The
unique existential quantifier _! is introduced by writing, for any formula __*x*_,_!*x*_ _*x*_ as
an abbreviation of the formula _*x*_ _ _*x*_ & _*y*_ _ _*y*_ _ *x* _ *y*__.*L*_#
will also allow the
formation of terms of the form _*x*|_ _*x*__, for any formula _ containing the free
variable *x*. Such terms are called non-classical sets; we shall use upper case letters
A,*B*, . . . for such sets. For each non-classical set *A* _ _*x*|_ _*x*__ the formulas
_*x*_*x* _ *A* _ _ _*x*__ and _*x*_*x* _ *A* _ _ _*x*,*A*__ is called the defining axioms for the
non-classical set *A*.
Remark 2.**1**.Remind that in logic 2*L*_#
with restricted modus ponens rule
the statement _ _ _ _ _ _ _ does not always guarantee that
_,_ _ _ _*RMP*_ _2. 1_
since for some _ and _ possible
_,_ _ _ *RMP*_ _2. 2_
even if the statement _ _ _ _ _ _ _ holds [8],see Appendix A.
Abbreviation 2.**1**We write for the sake of brevity instead (1.1) by
_ _*s*_ _2. 3_
and we often write instead (1.2) by
_ _*w*_. _2. 4_
Remark 2.**2**.Let *A* be an nonclassical set.Note that in set theory NC_#
the following
true formula

_*A*_*x*_*x* _ *A* _ _ _*x*,*A*__ _2. 5_
does not always guarantee that
x _ *A*, *x* _ *A* _ _ _*x*,*A*_ _*RMP*_ _*x*,*A*_ _2. 6_
even if *x* _ *A* holds and (or)
__*x*,*A*_,_ _*x*,*A*_ _ *x* _ *A* _*RMP* *x* _ *A*; _2. 7_
even __*x*,*A*_ holds, since for nonclassical set *A* for some *y* possible
y _ *A*, *y* _ *A* _ _ _*y*,*A*_ *RMP*_ _*y*,*A*_ _2. 8_
and (or)

__*y*,*A*_, __*y*,*A*_ _ *y* _ *A* *RMP* *y* _ *A*. _2. 9_

Remark 2.3.Note that in this paper the formulas

_*a*_*x*_*x* _ *a* _ __*x*_ _ *x* _ *u*_ _2. 10_

and more general formulas

_*a*_*x*_*x* _ *a* _ __*x*, *a*_ _ *x* _ *u*_ _2. 11_

is considered as the defining axioms for the classical set *a*.

Remark 2.4.Let *a* be a classical set. Note that in **NC**_#

: (i) the following true formula

_*a*_*x*_*x* _ *a* _ __*x*, *a*_ _ *x* _ *u*_ _2. 12_

always guarantee that

x _ *a*, *x* _ *a* _ __*x*, *a*_ _*RMP* __*x*_ _2. 13_

if *x* _ *a* holds and

__*x*_,__*x*_ _ *x* _ *a* _*RMP* *x* _ *a*; _2. 14_

if __*x*_ holds;

In order to emphasize this fact mentioned above in Remark 2.1-2.3,
we rewrite the defining axioms in general case for the nonclassical sets in the following
form

_*A*_*x*__*x* _ *A* _*s*__*x*,*A*__ _*x* _ *A* _*w*__*x*,*A*___ _2. 15_

and similarly we rewrite the defining axioms in general case for the classical sets in the
following form

_*a*_*x*_*x* _ *a* _*s*__*x*_ _ _*x* _ *u*__. _2. 16_

Abbreviation 2.2.We write instead (2.15):

_*x*__*x* _ *A* _*s,w*__*x*,*A*___ _2. 17_

Definition 2.1. (1) Let *A* be a nonclassical set defined by formula (2.17).
Assum that: (i) for some *y* statement __*y*_ and statement __*y*_ _ *y* _ *A* holds and
(ii) __*y*_,__*y*_ _ *y* _ *A* *RMP* *y* _ *A*, *y* _ *A*, *y* _ *A* _ __*y*_ *RMP* __*y*_.
Then we say that *y* is a weak member of non-classical set *A* and abbreviate *y* _*w* *A*.

Abbreviation 2.3. Let *A* be a nonclassical set defined by formula (2.17) We abbreviate *x* _*s,w* *A* if the following statement *x* _*s* *A* *x* _*w* *A* holds, i.e.

x _*s,w* *A* _*def* _*x* _*s* *A* *x* _*w* *A*_. _2. 18_

Definition 2.3. The ordered pair of two sets *u*, *v* is defined as usual by

u, *v* _ __*u*_, _*u*, *v*__. _2. 19_

Definition 2.4. We define the Cartesian product of two nonclassical sets *A* and *B* as usual by

A _*s,w* *B* _ _ *x*, *y*|*x* _*s,w* *A* _ *y* _*s,w* *B*_ _2. 20_

Definition 2.5. A binary relation between two nonclassical sets *A*,*B* is a subset *R* _*s,w* *A* _*s,w* *B*. We also write *aR*$_{s,w}$*b* for _ *a*, *b* __*s,w* *R*. The doman **dom**_*R*_ and the range **ran**_*R*_ of *R* are defined by

dom_*R*_ _ _*x*|_*y*_*xR*$_{s,w}$*y*__, **ran**_*R*_ _ _*y* : _*x*_*xR*$_{s,w}$*y*__. _2. 21_

Definition 2.6.A relation *F*$_{s,w}$ is a function, or map, written **Fun**_*F*$_{s,w}$_, if for each *a* _*s,w* **dom**_*F*_ there is a unique *b* for which *aF*$_{s,w}$*b*. This unique *b* is written *F*_*a*_ or *Fa*. We write *F*$_{s,w}$: *A* _ *B* for the assertion that *F*$_{s,w}$ s a function with **dom**_*F*$_{s,w}$_ _ *A* and **ran**_*F*$_{s,w}$_ _ *B*. In this case we write *a* *F*$_{s,w}$_*a*_ for *F*$_{s,w}$*a*.

Definition 2.7.The identity map $\mathbf{1}_A$ on *A* is the map *A* _ *A* given by *a* *a*.
If *X* _*s,w* *A*, the map *x* *x* : *X* _ *A* is called the insertion map of *X* into *A*.

Definition 2.8.If $F_{s,w} : A \_ B$ and $X \_{s,w} A$, the restriction $F_{s,w}|X$ of $F_{s,w}$ to X is the map $X \_ A$ given by x $F_{s,w}\_x\_$. If $Y \_{s,w} B$, the inverse image of Y under $F_{s,w}$ is the set
$F_{s,w}$
$\_1 \_Y\_ \_ \_x \_{s,w} A : F_{s,w}\_x\_ \_{s,w} Y\_. \_2.22\_$
Given two functions $F_{s,w} : A \_ B, G_{s,w} : B \_ C$, we define the composite function $G_{s,w} \_ F_{s,w} : A \_ C$ to be the function a $G_{s,w}\_F_{s,w}\_a\_\_$. If $F_{s,w} : A \_ A$, we write $F_{s,w}$
2
for $F_{s,w} \_ F_{s,w}, F_{s,w}$
3 for $F_{s,w} \_ F_{s,w} \_ F_{s,w}$ etc.
Definition 2.9.A function $F_{s,w} : A \_ B$ is said to be monic if for all $x, y \_{s,w} A, F_{s,w}\_x\_ \_ F_{s,w}\_y\_$ implies $x \_ y$, epi if for any $b \_{s,w} B$ there is $a \_{s,w} A$ for which $b \_ F_{s,w}\_a\_$, and bijective, or a bijection, if it is both monic and epi. It is easily shown that
$F_{s,w}$ is bijective if and only if $F_{s,w}$ has an inverse, that is, a map $G_{s,w} : B \_ A$ such that $F_{s,w} \_ G_{s,w} \_ \mathbf{1}_B$ and $G_{s,w} \_ F_{s,w} \_ \mathbf{1}_A$.
Definition 2.10.Two sets X and Y are said to be equipollent, and we write $X \_{s,w} Y$, if there is a bijection between them.
Definition 2.11.Suppose we are given two sets I, A and an epi map $F_{s,w} : I \_ A$. Then $A \_ \_F_{s,w}\_i\_|i \_ I\_$ and so, if, for each $i \_{s,w} I$, we write a_i for $F_{s,w}\_i\_$, then A can be presented in the form of an indexed set $\_a_i : i \_{s,w} I\_$. If A is presented as an indexed set of sets $\_X_i|i \_{s,w} I\_$, then we write $\_{i\_I} X_i$ and $\_{i\_I} X_i$ for $\_A$ and A, respectively.
Definition 2.12.The projection maps $\_1 : A \_{s,w} B \_ A$ and $\_2 : A \_{s,w} B \_ B$ are defined to be the maps $\_ a, b \_$ a and $\_ a, b \_ b$ respectively.
Definition 2.13.For sets A, B, the exponential B^A is defined to be the set of all functions from A to B.

Axiom of nonregularity

Remind that a non-empty set u is called regular iff $\_x\_x \_ \_ \_ \_\_y \_ x\_\_x \ y \_ \_\_\_$.
Let's investigate what it says: suppose there were a non-empty x such that

$\_\_y \_ x\_\_x \ y \_ \_\_$. For any $z_1 \_ x$ we would be able to get $z_2 \_ z_1 \ x$. Since $z_2 \_ x$ we would be able to get $z_3 \_ z_2 \ x$. The process continues forever:
$. . .\_ z_{n\_1} \_ z_n. . .\_ z_4 \_ z_3 \_ z_2 \_ z_1 \_ x$.Thus if we don't wish to rule out such an infinite regress we forced accept the following statement:
$\_x\_x \_ \_ \_ \_\_y \_ x\_\_x \ y \_ \_\_\_. \_2.23\_$

Axiom of hyperinfinity.

Definition 2.14.(i) A non-empty transitive non regular set u is a well formed non regular set iff:
(i) there is unique countable sequence $\_u_{n\_n\_1}$
$\_$ such that
$. . .\_ u_{n\_1} \_ u_n. . .\_ u_4 \_ u_3 \_ u_2 \_ u_1 \_ u, \_2.24\_$
(ii) for any $n \_ \_$ and any $u_{n\_1} \_ u_n$:
$u_n \_ u_{n\_1}$
$\_ , \_2.25\_$
where $a\_ \_ a \_ \_a\_$.
(ii) we define a function $a\_{\_k\_}$ inductively by $a\_{\_k\_1\_} \_ \_a\_{\_k\_} \_\_$.
Definition 2.15. Let u and w are well formed non regular sets. We write $w \_ u$ iff

for any n _ _
w _ u_n. _2. 26_
Definition 2.16. We say that an well formed non regular set u is infinite (or hyperfinite) hypernatural number iff:
(I) For any member w _ u one and only one of the following conditions are satified:
(i) w _ _ or
(ii) w _ u_n for some n _ _ or
(iii) w _ u.
(II) Let _u be a set _u _ _z|z _ u_, then by relation _ _ _ a set _u is densely ordered with no first element.
(III) _ _ u.
Definition 2.17. Assume u _ _#, then u is infinite (hypernatural) number if u _ _#_.
Axiom of hyperinfinity
There exists a set _# such that:
(i) _ _ _#,
(ii) if u _ _#_ then there exists infinite (hypernatural) number v such that v _ u,
(iii) if u _ _#_ then there exists infinite (hypernatural) number w such that for any n _ _ : u_n_ _ w,
(iv) set _#_ is patially ordered by relation _ _ _ with no first and no last element.

3.Hypernaturals _#.

In this section nonstandard arithmetic **A**# related to hypernaturals _# is considered axiomatically.

Axioms of the nonstandard arithmetic A# are:

Axiom of hyperinfinity
There exists unique set _# such that:

(i) _ _ _#
(ii) if u is infinite (hypernatural) number then there exists infinite (hypernatural) number v such that v _ u
(iii) if u is infinite hypernatural number then there exists infinite (hypernatural) number w such that u _ w
(iv) set _#_ is patially ordered by relation _ _ _ with no first and no last element.
Axioms of infite _-induction
(i)
_S_S _ __ _
n_
_n _ S _$_s$ n_ _ S_ _$_s$ S _ _ . _3. 1_
(ii) Let F_x_ be a wff of the set theory **NC**_#
#, then

_
n_
_F_n_ _$_s$ F_n_ ___ _$_s$ _n_n _ __F_n_. _3. 2_
Definition 3.1.(i) Let _ be a hypernatural such that _ _ _#_. Let _0,_ _ _ _# be a set such that _x_x _ _0,_ _ 0 _ x _ __ and let _0,_ be a set _0,_ _ _ _0,__ _.
(ii) Let _ _ _#_ and let __ _ _#be a set such that
_x_x _ __ _ _k_k _ 0__0 _ x _ __$_k$_ __. _3. 3_
Definition 3.2.Let F_x_ be a wff of **NC**_#

$_\#$ with unique free variable x.We will say that
a wff $F\_x\_$ is restricted on a classical set S such that $S\ \_s\ \_\#$ iff the following condition is satisfied

$\_\_\_\ \_\ \_\#\backslash S\ \_s\ \_F\_\_\_\_.\ \_3.\ 4\_$

Definition 3.3.Let $F\_x\_$ be a wff of $\mathbf{NC}\_\#$
$_\#$ with unique free variable x.We will say that
a wff $F\_x\_$ is strictly restricted on a set S such that $S\ \_s\ \_\#$ iff there is no proper subset $S\_\ \_\ S$ such that a wff $F\_x\_$ is restricted on a set $S\_$.

Example 3.1.(i)Let $\mathbf{fin}\_\_\_,\_\ \_\ \_\#$ be a wff formula such that $\mathbf{fin}\_\_\_\ \_s\_\ \_\ \_$.
Obviously wff $\mathbf{fin}\_\_\_$ is strictly restricted on a set _ since $\_\_\_\ \_\ \_\#\backslash\_\ \_s\ \_\mathbf{fin}\_\_\_\_$.
Let $\mathbf{hfin}\_\_\_,\_\ \_\ \_\#$ be a wff formula such that $\mathbf{hfin}\_\_\_\ \_s\_\ \_\ \_\#\backslash\_$ since
$\_\_\_\ \_\ \_\ \_s\ \_\mathbf{hfin}\_\_\_\_$.

Definition 3.4. Let $F\_x\_$ be a wff of $\mathbf{NC}\_\#$
$_\#$ with unique free variable x.We will say that a
wff $F\_x\_$ is unrestricted if wff $F\_x\_$ is not restricted on any set S such that $S\ \_\ \_\#$.

Axiom of hyperfinite induction 1

$\_S\_S\ \_s\ \_0,\_\_\_\_\_\_\ \_s\ \_\#\ \_$

$\_\_\_\ \_s\ \_0,\_\_\_\ \_$

$0\_\_\_$

$\_\_\ \_s S\ \_\_\_\_s S\_\ \_s S\ \_\ \_0,\_\_\ .$

$\_3.\ 5\_$

Axiom of hyperfinite induction 1_

$\_S\_S\ \_s\ \_0,\_\_\_\_\_\_\ \_\ \_\#\ \_$

$\_\_\_\ \_\ \_0,\_\_\_\ \_$

$0\_\_\_$

$\_\_\ \_S\ \_\_\_\ S\_\ \_\ S\ \_\ \_0,\_\_\ .$

$\_3.\ 6\_$

Axiom of hyper infinite induction 1$\_S\_S\ \_s\ \_\#\ \_\ \_\_\_\ \_\ \_\#\ \_\ \_$

$0\_\_\_$

$\_\_\ \_s S\ \_\_\_\_s S\_\ \_s S\ \_s\ \_\#\ .\ \_3.\ 7\_$

Definition 3.5.A set $S\ \_s\ \_\#$ is a hyper inductive if the following statement holds

$\_$

$\_\_\#$

$\_\_\ \_s S\ \_s\_\_\ \_s S\_.\ \_3.\ 8\_$

Obviously a set $\_\#$ is a hyper inductive. Thus axiom of hyper infinite induction 1 asserts that a set $\_\#$ this is the smallest hyper inductive set.

Axioms of hyperfinite induction 2

Let $F\_x\_$ be a wff of the set theory $\mathbf{NC}\_\#$
$_\#$ strictly restricted on a set _0,_ _ then

$\_\_\_\ \_\ \_0,\_\_\_\ \_$

$0\_\_\_$

$\_F\_\_\ \_s F\_\_\ \_\_\ \_s\ \_\_\_\ \_\ \_0,\_\_F\_\_.\ \_3.\ 9\_$

Let $F\_x\_$ be a wff of the set theory $\mathbf{NC}\_\#$
$_\#$ strictly restricted on a set _0,_ _ _ then

$\_\_\_\ \_\ \_0,\_\_\_\ \_$

$0\_\_\_$

$\_F\_\_\ \_s F\_\_\ \_\_\ \_s\ \_\_\_\ \_\ \_0,\_\_\_F\_\_.\ \_3.\ 10\_$

Axiom of hyper infinite induction 2

Let $F\_x\_$ be anrestricted wff of the set theory NC$_\#$

then

__ _ _ _ _ _# _ _
0_ _ _
_F_ _ _ _s F_ _ _ ___ _s _ _ _ _ _ _# _F_ _ _. _3. 11_

The main restricted rules of conclusion.

If $\mathbf{A}_{\#}$ _ A then _A $_{RMP}$ B,where B _ _#.

Thus if statement A holds in $\mathbf{A}_{\#}$ we cannot obtain from _A by restricted rules of conclusion any formula B _ _# whatsoever.

The Generalized Recursion Theorem.

Theorem 3.1. Let S be a set, c _ S and $G : S$ _ S is any function with **dom**_G_ _ S and **range**_G_ _ S.Let W_G_ _ _#_ S be a binary relation such that:
(a) _1, c_ _ W_G_ and (b) if _x, y_ _ W_G_ then _**Sc**_x_,G_y__ _ W_G_.
Then there exists a function _ : _# _ S such that: (i) **dom**___ _ _#and **range**___ _ S; (ii) __1_ _ c; (iii) for all x _ _#, __**Sc**_x__ _ G___x__.

1.The desired function _ is a certain relation **W** _ _#_ S. It is to have the properties:
(ii_) _1, c_ _ **W**; (iii_) if _x, y_ _ **W** then _**Sc**_x_,G_y__ _ **W**.

Remark 3.1. The latter is just another way of expressing (iii), that if
__x_ _ y _3. 12_
then
__**Sc**_x__ _ G_y_. _3. 13_

Remark 3.2.Note that any relation **W** mentioned above is hyper inductive relation since the hyper inductivity conditions (ii_)-(iii_) are satisfied.
However there are many hyper inductive relations which satisfy the conditions (ii_)-(iii_); on such is _#_ S.What distinguishes the desired function from all these other relations is that we want _a, b_ to be on it only as required by (ii_) and (iii_). In other words, it is to be the smallest relation satisfying(ii_)-(iii_). This can be expressed precisely as follows:
(1) Let **M** be a set of the relations **W** satisfying the conditions (ii_) and (iii_);

then we define a set _ _ __
W_M
W. Hence (2) whenever **W** _ **M** then _ _ **W**.
We shall now show that we can derived from (1) that _ is also one relation in **M**.
(3) _1, c_ _ _.

This follows immediately from the definition of __
W_M
and the fact that _1, c_ _ **W** for all
W _ **M**. (4) If _x, y_ _ _ then _**Sc**_x_,G_y__ _ _.
For if _x, y_ _ _ then _x, y_ _ **W** for all **W** _ **M**;hence by (iii_)
_**Sc**_x_,G_y__ _ **W** for all **W** _ **M** so that _**Sc**_x_,G_y__ _ _ by (1).
We must now verify that _ ís actually a function, i,e., we wish to show that for any x, z_1, z_2 _ _#, if _x, z_1_ _ _ and _x, z_2_ _ _, then z_1 _ z_2.
We shall prove this by hyper infinite induction on x. Let
(5) A _ _x|x _ _# and for all z_1, z_2 _ _#, if _x, z_1_ _ _ and _x, z_2_ _ _ then z_1 _ z_2_.
We shall show A _ _# by applying hyper infinite induction. First we have (6) 1 _ A.
To prove (6), it suffices to show that for any z, if _1, z_ _ _ then z _ c.
We prove this by contradiction; in other words, suppose to tbe contrary that there is some z with _1, z_ _ _ but z _ c. Consider the relation W _ ___1, z__. Since

_1, *c*_ _ _ and _1, *c*_ _ _1, *z*_, it follows that _1, *c*_ _ *W*. Moreover, whenever _*u*, *y*_ _ *W* then _*u*, *y*_ _ _ and hence _**Sc**_*u*_,*G*_*y*__ _ _ but **Sc**_*u*_ _ 1, so _**Sc**_*u*_,*G*_*y*__ _ _1, *z*_, and hence _**Sc**_*u*_,*G*_*y*__ _ *W*. Thus *W* satisfies both conditions (ii_) and (iii_); in other words, **W** _ **M**. But then it follows from (2) that _ _ **W** however this is elearly false sinee _1, *z*_ _ _ and _1, *z*_ _ **W**. Tbus our hypothesis has led us to a contradiction, and henee (6) is proved. Next we show that

(7) for any *x* _ _# if *x* _ *A* then **Sc**_*x*_ _ *A*.

Suppose that *x* _ *A*, so that whenever _*x*, z_1_ _ _ and _*x*, z_2_ _ _ then z_1 _ z_2. We must show that whenever _**Sc**_*x*_,w_1_ _ _and _**Sc**_*x*_,w_2_ _ _ then w_1 _ w_2. To prove this, it suffices to show that

(8) whenever _**Sc**_*x*_,*w*_ _ _ then there exists a *z* with *w* _ *G*_*z*_ and _*x*, *z*_ _ _.

For if (8) ia true, we would have for the given w_1,w_2 some z_1 _ z_2 with w_1 _ *G*_z_1_, w_2 _ *G*_z_2_, _*x*, z_1_ _ _ and _*x*, z_2_ _ _. Then, since *x* _ *A*, z_1 _ z_2 and henee *G*_z_1_ _ *G*_z_2_ , that is, w_1 _ w_2.

Now to prove (8) suppose, to the contrary, that it is not true; in other words, suppose that we have some *w* with _**Sc**_*x*_,*w*_ _ _ but such that for all *z* which _*x*, *z*_ _ _ we have *w* _ *G*_*z*_. Consider the relation **W** _ ___**Sc**_*x*_,*w*__. We shall show that **W** _ **M**. First of all _1, *c*_ _ _and _1, *c*_ _ _**Sc**_*x*_,*w*_; hence _1, *c*_ _ **W**. Suppose tbat _*u*, *y*_ _ **W**; then _*u*, *y*_ _ _ and _**Sc**_*u*_,*G*_*y*__ _ _. Clearly if *u* _ *x* then _**Sc**_*u*_,*G*_*y*__ _ _**Sc**_*x*_,*w*_, so that in this case _*Sc*_*u*_,*G*_*y*__ _ **W**. On the other hand, if *u* _ *x* and _**Sc**_*u*_,*G*_*y*__ _ _**Sc**_*x*_,*w*_, then *w* _ *G*_*y*_, where _*x*, *y*_ _ _, contrary to the choice of *w* henee _**Sc**_*u*_,*G*_*y*__ _ _**Sc**_*x*_,*w*__, so again _**Sc**_*u*_,*G*_*y*__ _ **W**. Thus whenever _*u*, *y*_ _ **W**, also _**Sc**_*u*_,*G*_*y*__ _ **W**. Now that we have shown **W** _ **M** we see by (2) that _ _ **W** but this is false since _**Sc**_*x*_,*w*_ _ _ and _**Sc**_*x*_,*w*_ _ **W**. Thus our hypothesis that (8) is incorrect has led to a contradiction, and now (8) is proved. Sinee (7) follows from (8), we have by hyper infinite induction from (6) that *A* _ _#. Hence (9) _ is a function. We have still to prove that _ satisfies,condition (i); we must show that

for each *x* _ _# there· is a *y* with _*x*, *y*_ _ _. Since _ _ _# _ *S*, it will then follow that **dom**___ _ _# and **range**___ _ *S*. Let *B* _ **dom**___, that is,

(10) *B* _ _*x*|*x* _ _# and for some *y*, _*x*, *y*_ _ __.

We prove now by hyper infinite induction that *B* _ _#. First, 1 _ *B*, sínce _1, *c*_ _ _ by (3). Next, if *x* _ *B*, pick some *y* with _*x*, *y*_ _ _; then by (4), _**Sc**_*x*_,*G*_*y*__ _ _, and henee **Sc**_*x*_ _ *B*.

Thus concludes the first part of the proof, that there is at least one function _ satisfying conditions (i)-(iii).

Part 2. We prove that there cannot be more than one such function.

Suppose that _₁ and _₂ both satisfy the conditions (i)-(iii) we wish to show _₁_ _₂, i.e., that for all *x* _ _#,_₁_*x*_ _ _₂_*x*_. Thus is proved by hyper infinite induction on *X*. By (ii), _₁_1_ _ *c* and _₂_1_ _ *c*, so _₁_1_ _ _₂_1_. Suppose that _₁_*x*_ _ _₂_*x*_; then _₁_**Sc**_*x*__ _ *G*__₁_*x*__ and _₂_**Sc**_*x*__ _ *G*__₂_*x*__, so _₁_**Sc**_*x*__ _ _₂_**Sc**_*x*__.

Theorem 3.2. Let *S* be a set, *c* _ *S* and *G* : *S* _ _# _ *S* is a binary function with **dom**_*G*_ _ *S* _ _# and **range**_*G*_ _ *S*.

Then there exists a function _ : _# _ *S* such that:

(i) **dom**___ _ _#and **range**___ _ *S*; (ii) __1_ _ *c*; (iii) for all *x* _ _# , __**Sc**_*x*__ _ *G*___*x*_, *x*_.

We omit the proof of the Theorem 3.2 since it can be given by simple modification of the proof to Theorem 3.1.

4. Nonconservative extension of the model theoretical NSA

Remind that Robinson nonstandard analysis (RNA) many developed using set-theoretical objects called superstructures [14]-[17]. A superstructure **V**_S_ over a set S is a set defined by the following way:

V0_S_ _ S,**V**n_1_S_ _ **V**n_S_ _ _P_**V**n_S__,**V**_S_ _ _ n__
Vn_S_. _4. 1_

Superstructures of the empty set consist of sets of infinite rank in the cumulative hierarchy and therefore do not satisfy the in_nity axiom. Making S _ _ will suffice for virtually any construction necessary in analysis.

Bounded formulas are formulas where all quantifiers occur in the form
_x_x _ y _ _,_x_x _ y _ _. _4. 2_

A nonstandard embedding is a mapping
_ : **V**_X_ _ **V**_Y_ _4. 3_

from a superstructure **V**_X_ called the standard universum, into another superstructure **V**_Y_, called nonstandard universum, satisfying the following postulates:

1. Y _ _X

2.**Transfer Principle**.For every bounded formula __x_1, . . . , x_n_ and elements a_1, . . . , a_n _ **V**_X_, the property _ is true for a_1, . . . , a_n in the standard universum if and only if it is true for _a_1 , . . . , _a_n in the nonstandard universum: **V**_X_,_ _ __a_1, . . . , a_n _ _ **V**_Y_,_ _ __a_1 , . . . , _a_n _. _4. 4_

3.**Non-triviality**.For every infinite set A in the standard universum, the set __a|a _ A_ is a proper subset of _A.

Definition 4.1.[17] A set x is internal if and only if x is an element of _A for some element A of **V**___. Let X be a set with A _ _A_i_i_I a family of subsets of X. Then the collection A has the infinite intersection property, if any infinite subcollection J _ I has non-empty intersection. Nonstandard universum is _-saturated if whenever _A_i_i_$I$is a collection of internal sets with the infinite intersection property and the cardinality of I is less than or equal to _,__
i_I
A_i _ _.

Remark 4.1.Remind that: (i) for each standard universum U _ **V**_X_ there exists canonical language _ _ _U, (ii) for each nonstandard universum W _ **V**_Y_ there exists corresponding canonical nonstandard language _ _ _ _W[17].

3_.The restricted rules of conclusion.
If W _ A then _A B,where B _ _ _ B _ __.
Thus if A holds in W we cannot obtain from _A any formula B whatsoever.

Remark 4.2. We write _ _ A instead W _ A.

Definition 4.2.[6]-[7].A set S _ _is a hyper inductive if the following statement holds

_
___
__ _ S _s__ _ S_, _4. 5_

where __ _ _ _ 1.Obviously a set __ is a hyper inductive. As we see later there is just one hyper inductive subset of __,namely __itself.

In this paper we apply the following hyper inductive definitions of a sets [6]-[7]

_ S_ _ _ _ S _s _
0_ _ _
_ _ _ S _s_ _ _ S_ , _4. 6_
We extend up Robinson nonstandard analysis (**RNA**) by adding the following postulate:
4.Any hyper inductive set S is internal.
Remark 4.1.The statement **4** is not provable in ZFC but provable in a set theory **NC**_#
,
see.Thus postulates 1-4 gives an nonconservative extension of RNA and we denote such extension by **NERNA**.
Remark 4.2.Note that NERNA of course based on the same gyper infinitary logic with Restricted Modus Ponens Rule as a set theory **NC**_#
.
Remind that in RNA the following induction principle holds.
Theorem 4.1.[17]. Assume that S _ _ is internal set, then
_1 _ S_ _ _x_x _ S _ x _ 1_ _ S _ _. _4. 7_
In NERNA Theorem 4.1also holds.
Remark 4.3.It follows from postulate 4 and Theorem 3.1 that any hyper inductive set S is equivalent to _ : S _ _.
Remark 4.4. Note that the following statement is provable in **NC**_#
:
4_ Axiom of hyper infinite induction_S_S _ _ _ _ _ _ _ _ _
0_ _ _
_ _ _ S _ _ _ S_ _ S _ _ . _4. 8_
Thus postulate **4 of the theory** NERNA is provable in **NC**_#
.

Rules of conclusion

MRR (Main Restricted rule of conclusion)
Let _ _x_ be a wff with one free variable x and such that _n _ __ _ **V**_Y_ _ _n_, then_ _n_ B, i.e., if statement A holds in **V**_Y_ we cannot obtain from _A any formula B whatsoever.
Remark 4.5.The MRR is necessarily in natural way, since by assumption _ _n_ one obtains directly the apparent contradiction _ _n_ _ _ _n_ from which by unrestricted modus ponens rule (UMPR) one obtains _ _n_ _ _ _n_ _$_{UMPR}$ B.
Example 4.1. Remind the proof of the following statement: structure __,__ is a well-ordered set.
Proof.Let X be a nonempty subset of _. Suppose X does not have a _ -least element. Then consider the set _\X.
Case (1) _\X _ _. Then X _ _ and so 0 is a _ -least element. Contradiction.
Case (2) _\$X$ _ _.Then 1 _ _\$X$ otherwise 1 is a _ -least element. Contradiction.
Case (3) _\$X$ _ _. Assume now that there exists an n _ _\$X$ such that n _ 1.
Since we have supposed that X does not have a least element, thus n _ 1 _ X.
Thus we see that for all n : n _ _\$X$ implies that n _ 1 _ _\$X$. We can conclude by induction that n _ _\$X$ for all n _ _. Thus _\$X$ _ _ implies X _ _.
This is a contradiction to X being a nonempty subset of _.
We set now X_1 _ __, thus _\$X_1$ _ _. In contrast with a set X the assumption n _ _\$X_1$ implies that n _ 1 _ _\$X_1$ if and only if n is finite, since for any infinite n _ __ the assumption n _ _\$X_1$ contradicts with a true statement **V**_Y_ _ n _ _\$X_1$ _ _ and therefore in accordance with MRR we cannot obtain

from n _ __\X_1 any formula B whatsoever.

5.IST# and BST#

The axiomatics IST (Internal Set Theory) was presented in 1977 [11] and in a sense formulates within first-order language the behaviour of standard and internal sets of a nonstandard model of ZFC. This were done by adding the unary standardness predicate "st" to the language of ZFC as well as adding to the axioms of ZFC three new axiom schemes involving the predicate "st": **Idealization**, **Standardization** and **Transfer**.

Remark 5.1.Formulas which do not use the predicate **st** are called internal formulas (or _-formulas) and formulas that use this new predicate are called external formulas (or st-_-formulas).A formula _ is standard if only standard constants occur in _.

Abbreviaion 5.1.We denote a set of the all naturals by _#and a set of the all finite naturals by _.

Abbreviaion 5.2.We write **fin**_x_ meaning 'x is finite'. Let _ _x_ be a **st**- _ -formula:

1._st x_ _x_ abbreviates _x_**st**_x_ _ _ _x__.2._st x_ _x_ abbreviates _x_**st**_x_ _ _ _x__.

3._fin$x$_ _x_ abbreviates _x_**fin**_x__ _ _ _x__.4._fin x_ _x_ abbreviates _x_**fin**_x_ _ _ _x__.

5._stfin x_ _x_ abbreviates _x_**st**_x_ _ **fin**_x__ _ _ _x__.

6._stfin x_ _x_ abbreviates _x_**st**_x_ _ **fin**_x_ _ _ _x__.7._s

stfin x_ _x_ abbreviates _x_**st**_x_ _ **fin**_x__ _s_ _x__.

8._s,w

stfin x_ _x_ abbreviates _x_**st**_x_ _ **fin**_x__ _s,w_ _x__.

9._stfin x_ _x_ abbreviates _x_**st**_x_ _ **fin**_x_ _ _ _x__.

The fundamental axioms of **IST#** :

(**I**) **Idealization for classical sets**

_s

stfin$F$CL_yCL_xCL _s F_RCL_x, y_ _s _bCL_s

stxRCL_x, b__ _5. 7_

for any internal classical relation RCL_x, y_.

Remark 5.5.The idealization axiom obviously states that saying that for any fixed classical finite set F there is a classical y such that RCL_x, y_ holds for all classical x _s F is the same as saying that there is a classical b such that for all fixed classical x the classical relation RCL_x, b_ holds.

(**II**) **Standardization for classical sets**

_st$A$CL_st$B$CL_st$x$CL_x _ B _s x _ A _ _ _x__ _5. 8_

for every st-_-formula _ with arbitrary (internal) parameters.

(**III**) **Transfer for classical sets**

_sty_1

CL, . . . , y_n

CL_st$x$CL_ _ _$x, y_1, \ldots, y_n$__ _s _xCL_ _$x, y_1, \ldots, y_n$_ _5. 9_

for all internal formulas _ _$x, y_1, \ldots, y_n$_.

Boundedness

_xCL_st$y$CL_x _s y_ _5. 10_

and since (5.10) contradicts idealization the following (bounded) form is taken instead:

(**IV**) **Bounded Idealization for classical sets**

For every _-formula R :

_st$Y$CL__stfin$F$CL_yCL _ Y__xCL_x _ F_R_x, y_ _s _bCL_b _ Y__stxR_x, b___. _5. 11_

(**V**) **Idealization for nonclassical sets**

_s,w

stfin$F$NCL_yNCL_xNCL _$_{s,w}$ F_RNCL_x, y_ _$_{s,w}$ _bNCL_$_{s,w}$
st xRNCL_x, b__ _5. 12_
for any internal nonclassical relation RNCL_x, y_.
Remark 5.6.The idealization axiom obviously states that saying that for any fixed nonclassical finite set F there is a classical y such that RNCL_x, y_ holds for all classical x _$_s$ F is the same as saying that there is a classical b such that for all fixed classical x the nonclassical relation RNCL_x, b_ holds.
(**VI**) **Standardization for nonclassical sets**
_$_{s,w}$
st ANCL_st$B$NCL_$_{s,w}$
st xNCL_x _$_{s,w}$ B _$_{s,w}$ x _$_{s,w}$ A _ _ _x__ _5. 13_
for every st-_$_{s,w}$-formula _ with arbitrary (internal) parameters.
(**VII**) **Transfer for nonclassical sets**
_$_{s,w}$
st y_1
NCL, . . . , y_n
NCL_st$x$NCL_ _ _$x, y_1, . . . , y_n$__ _$_{s,w}$ _$_{s,w}$$x$NCL_ _$x, y_1, . . . , y_n$_ _5. 14_
for all internal _ _$x, y_1, . . . , y_n$_.
Boundedness for nonclassical sets
_$_{s,w}$$xNCL_sty$NCL_$x$ _$_{s,w}$ y_ _5. 15_
and since (5.15) contradicts idealization the following (bounded) form is taken instead:
(**VIII**) **Bounded Idealization for nonclassical sets**For every _$_{s,w}$-formula R :
_$_{s,w}$
st YNCL _$_{s,w}$
stfin$F$NCL_yNCL _$_{s,w}$ Y__$_{s,w}$$x$NCL_$x$ _ F_R_x, y_ _$_{s,w}$
bNCLb _ Y__$_{s,w}$
st xR_x, b___.
_5. 16_
(**IX**) **Internal Induction**
_S_S _$_s$ _$_\#$ _ _ _ _ _ _ _$_\#$ _ _
0_ _ _
_ _ _$_s$ S _ _ _ _$_s$ S_ _$_s$ S _$_s$ _$_\#$. _5. 17_
The main restricted rules of conclusion.
If $\mathbf{IST}_\#$ _ A then _A B,where B _ _$_\#$.
Thus if statement A holds in $\mathbf{IST}_\#$ we cannot obtain from _A any formula B whatsoever.

6.External Set Theory HST$_\#$.

6.1.External Set Theory HST.

A "perfect" external set theory (a nonstandard set theory that includes external sets) should satisfy some requirements:
(**I**) It should be a conservative extension of classical mathematics (usually ZFC) so that all classical mathematical theorems and constructions remain valid.
(**II**)The theory should also allow to perform nonstandard constructions in its full generality and therefore include a strong version of saturation (called idealization in IST and bounded idealization in BST) and transfer principles.
(**III**) Finally it should allow to build, for any given

set, the standard set of all its standard elements. This is called standardization. This means that ideally it should be something like an extension of IST allowing external sets and quantification over external formulas. However, as pointed out by Hrbácek [10] such a theory cannot exist. In fact, the axiom of regularity cannot be extended to the external universe. To see that let _ _ # denote the external set of infinitely large real numbers. Observe that for all _ in the (nonempty) external set _ _ # _, one has _ _ # _ _ _ _ . Additionally, if one wishes to formulate a nonstandard set theory with IST-style saturation 4 , the replacement axiom in the external universe contradicts both power set and choice. Let n be a nonstandard natural number. By saturation there is a 1-1 embedding into n, for all ordinals. So by power set and transfer the class Ord is a set (see Theorem 1.3.9 and Remark 1.3.10 in [13]).

Remark 6.1.To be of standard size means to be an image of the set of all standard elements of a standard set (In HST,a set X is standard size if and only if X is well-ordered) .To see that choice fails, let x be well-ordered by a relation _. Consider the class of all standard ordinals $_Ord$, well-ordered by _. We use the theorem that whenever two sets are well-ordered there is an order preserving embedding of one into the other. Clearly $_Ord$ cannot be embedded into x,otherwise $_Ord$ would be a set. Then there is an embedding of x into Ord. In fact, to an initial segment of $_Ord$. This means that x is of standard size. **Remark 6.2**. As a consequence, sets which are not of standard size cannot be well-ordered (see Theorem 1.3.1 in [13]). These results are known as the Hrbácek's paradoxes.

The first problem is not in fact a "real" problem because the regularity axiom is given so that every set is obtained at some level of the cumulative hierarchy over _ as mentioned above and has no great impact on which theorems are true. This "nice picture" of the universe is contested by some mathematicians and a suitable anti-foundation axiom can be taken instead,see for example [18].

In [12] Hrbácek considered already two possibilities to avoid this. The first one was to lose both power set and choice for external sets, leading to the system NS_1 . The second one was to lose the replacement axiom for external sets, which lead to his theory NS_2

A third possibility was developed by Kanovei [13].The idea is to restrict saturation by a standard infinite cardinal in order to reinstate the power set axiom. This is a system of partially saturated external sets which modifies the system HST (described below), called HST_k . This may be a solution for many practical purposes but not a solution as a foundational system for the nonstandard methods.

The theory BST possesses an extension to HST, which formulates within first-order language essential aspects of the behaviour of standard, internal and external sets within a nonstandard model, much as in Hrbácek's system NS_1 . The system HST is conservative over ZFC and equiconsistent with both BST and ZFC.

A set in HST is called internal if it is element of a standard set (see also the "Boundedness" axiom).

Remark 6.3. Below we use (definable) classes, they only should be interpreted as abbreviations of formulas with sets. Two important definable classes in HST are the class of standard sets

S _ _*x*|**st**_*x*__ _6. 1_
and the class of internal sets
I _ *x*|_*y*_*st*_*y*_ _ *x* _ *y*_ _6. 2_

6.2.HST Axioms

(**I**) **Axioms for all sets**.
The axioms of this group are valid for all sets. These axioms are similar to the respective ones of ZFC with the diference that in HST they are presented in the full language. This implies in particular, by the axiom of separation, that the theory HST deals with external sets; for example if *X* is standard and infinite, then _*x* _ *X*|**st**_*x*__ is an external set.

1.**Extensionality**
_*X*_*Y*__*x*_*x* _ *X* _ *x* _ *Y*_ _ *X* _ *Y*_.

2.**Pair**
_*a*_*b*_*A*_*x*_*x* _ *A* _ _*x* _ *a* *x* _ *b*__.

3.**Union**
_*A*_*B*_*x*_*x* _ *B* _ _*X* _ *A*_*x* _ *X*__.**4**.**Infinity**
_*X*__ _ *X* _ _*x*_*x* _ *X* _ _*x* _ _*x*_ _ *X*__.

5.**Separation**
_*X*_*Y*_*x*_*x* _ *Y* _ _*x* _ *X* _ _*x*___.

6.**Collection**
_*X*_*Y*_*x* _ *X*__*y*_*x*, *y*__ _ _*y* _ *Y*__*x*, *y*__.

The power set, regularity and choice axioms of ZFC are not valid in general. This is because, as mentioned above, each one of these axioms (if considered in the full language of HST) leads to a contradiction.

(**II**) **Axioms for standard and internal sets**
In this group as well as in the next there are axioms which are not valid for all sets. The first axiom scheme states that all ZFC axioms, when restricted to standard parameters are valid in HST

1. *ZFC*st.
This means, in particular, that the following are axioms of HST:

(a) **Regularity**st
_st*S*_*S* _ _ _ __st*x* _ *S*_ _ _*x* *S* _ ____ _6. 1_

(b) **Power Set**st
_st*X*_st*Y*_st*x*_*x* _ *Y* _ *x* _ *X*_ _6. 2_

(c) **Choice**st
_st*S*_st*Y*_st*x*_*x* _ *S*_______st*z*_*Y* *x* _ _*z*____ _6. 3_

The fact that every axiom of *ZFC* restricted to standard sets is an axiom of *HST* means that the class *S* models *ZFC*.

2.**Transfer**
_st*x*1, . . . ,_st*xn*_ _ _*x*1, . . . , *xn* ___ _ _int_*x*1, . . . , *xn* _ _6. 4_
where _ is an arbitrary closed _- formula containing only standard parameters
This means that the universe **I** is an elementary extension of **S** in the *ZFC* language.

3. **Transitivity of I**
_int*x*_*y*_*y* _ *x* _ **int**_*y*__ _6. 5_
The next axiom states that the class **I** is regular. This means that sets in *HST* are built over **I** in a way similar to the Von Neumann hierarchy of sets in *ZFC* over _.

4. **Regularity over I**
_*X* _ __*x* _ *X*_*x* *X* _ **I**_. _6. 6_

5. **Standardization**

_X_st$Y$_X **S** _ Y **S**_. _6. 7_

This axiom implies that the only sets consisting entirely of standard sets are of the form Y **S**, where Y _ **S**.

Axioms for sets of standard size

1. **Saturation**

If A _ **I** is a standard size set then

___X,Y _ A_ _ X Y _ A_ _ _X _ A_ _ X _ _____ _ A _ _. _6. 8_**2**. **Standard Size Choice**

Choice is available in the case where the domain of the choice function is of standard size.Let X be a set of standard size and F a function on X. Then

_x _ X__F_x_ _ ____ _ _f_f_x_ _ F_x____ _6. 9_

3. **Dependent Choice**

Any nonempty partially ordered set without maximal elements includes a nonempty linearly ordered subset (sequence) where any element has its immediate successor.

6.3.Nonconservative extension of the HST. External Set Theory HST#.

(**I**) **Axioms for all sets**.

The axioms of this group are valid for all sets. These axioms are similar to the respective ones of **NC**_#

\# with the diference that in HST# they are presented in the full language. This implies in particular, by the axiom of separation, that the theory HST# deals with external sets; for example if X is standard and infinite classical set, then _x _ XCL|**st**_x__ is an external classical set of the set theory **NC**_#

\#**st**.

This means, in particular, that the following are axioms of HST#:

I.Axioms for a classcal sets

(a) **Regularity$_{st}$ for a classcal sets**

_st$S$_S _ _ _s ___st$x$ _s S_ _ _x s S _ ____ _6. 10_

(b) **Power Set$_{st}$ for a classcal sets**

_st$X$_st$Y$_st$x$_x _ Y _s x _ X_ _6. 11_

(c) **Choice$_{st}$ for a classcal sets**

_st$S$_st$Y$_st$x$_x _s S\$s$________st$z$_$Y$ s x _ _z____ _6. 12_

(d) **Transfer for a classcal sets**

_stx_1, . . . ,_stx_n_ _ _x_1, . . . , x_n ___ _ _int_x_1, . . . , x_n _ _6. 13_

where _ is an arbitrary closed _- formula containing only standard parameters

This means that the universe **I** is an elementary extension of **S** in the **NC**_#

\#**st** language.

(e) **Transitivity of I for a classcal sets**

int$x$ y_y _s x _s **int**_y__. _6. 14_

The next axiom states that the class **I** is regular. This means that sets in HST# are built over **I** in a way similar to the Von Neumann hierarchy of sets in **NC**_#

\#**st** over _.

(f) **Regularity over I for a classcal sets**

_X _ __x _s X_x s X _s **I**_. _6. 15_

(g) **Standardization for a classcal sets**

$\_X\_{\mathbf{st}}Y\_X\,{}_s\,\mathbf{S}\ \_\ Y\,{}_s\,\mathbf{S}\_.\ \_6.16\_$

This axiom implies that the only sets consisting entirely of standard sets are of the form $Y\,{}_s\,\mathbf{S}$, where $Y\ \_{}_s\,\mathbf{S}$. **Axioms for a classcal sets sets of standard size**

1. **Saturation for a classcal sets**

If $A\ \_{}_s\,\mathbf{I}$ is a standard size set then

$\_\_\_X,Y\ \_{}_s A\_\ \_{}_s X\,{}_s\,Y\ \_{}_s A\_\ \_\ \_X\ \_{}_s A\_\ \_{}_s X\ \_\ \_\_\_\_\_\ \_{}_{s\,s} A\ \_\ \_.\ \_6.17\_$

2. **Standard Size Choice for a classcal sets**

Choice is available in the case where the domain of the choice function is of standard size.Let X be a set of standard size and F a function on X. Then

$\_x\ \_{}_s X\_\_F\_x\_\ \_\ \_\_\_\_\ \_{}_s\ \_f\_f\_x\_\ \_{}_s F\_x\_\_\_\_\ \_6.18\_$

3. **Dependent Choice for a classcal sets**

Any nonempty partially ordered set without maximal elements includes a nonempty linearly ordered subset (sequence) where any $\_{}_{s,w}$ element has its immediate successor.

II.**Axioms for a nonclassical sets**

(a) **Regularity$_{\mathbf{st}}$ for a nonclassscal sets**

$\_{\mathbf{st}}S\mathrm{NCL}\_S\ \_\ \_\ \_{}_{s,w}\ \_\_\_{\mathbf{st}}x\ \_{}_{s,w}S\_\ \_\ \_x\,{}_{s,w}S\ \_\ \_\_\_\_\ \_6.19\_$

(b) **Power Set$_{\mathbf{st}}$ for a nonclasscal sets**

$\_{\mathbf{st}}X\mathrm{NCL}\_{\mathbf{st}}Y\mathrm{NCL}\_{\mathbf{st}}x\mathrm{NCL}\_x\ \_{}_{s,w}Y\ \_{}_{s,w}x\ \_{}_{s,w}X\_\ \_6.20\_$

(c) **Choice$_{\mathbf{st}}$ for a a nonclasscal sets**

$\_{\mathbf{st}}S\_{\mathbf{st}}Y\_{\mathbf{st}}x\_x\ \_{}_{s,w}S\backslash s\_\_\_\_\_\_\_{\mathbf{st}}z\_Y\,{}_{s,w}x\ \_\ \_z\_\_\_\_\ \_6.21\_$

(d) **Transfer for a nonclasscal sets**

$\_{\mathbf{st}}x_1,\ldots,\_{\mathbf{st}}x_n\_\_\_x_1,\ldots,x_n\ \_\_\_\ \_{}_{s,w}\_\mathbf{int}\_x_1,\ldots,x_n\ \_\ \_6.22\_$

where _ is an arbitrary closed _- formula containing only standard parameters This means that the universe **I** is an elementary extension of **S** in the $\mathbf{NC}\_{}_{\#}$ $_{\#\mathbf{st}}$ language.

(e) **Transitivity of I for a nonclasscal sets**

$\_{\mathbf{int}}x\_y\_y\ \_{}_{s,w}x\ \_{}_{s,w}\mathbf{int}\_y\_\_\ \_6.23\_$

The next axiom states that the class **I** is regular. This means that sets in HST$_{\#}$ are built

over **I** in a way similar to the Von Neumann hierarchy of sets in $\mathbf{NC}\_{}_{\#}$ $_{\#\mathbf{st}}$ over _.

(f) **Regularity over I for a nonclasscal sets**

$\_X\ \_\ \_\_x\ \_{}_{s,w}X\_x\,{}_{s,w}X\ \_{}_{s,w}\mathbf{I}\_.\ \_6.24\_$

(g) **Standardization for a nonclasscal sets**

$\_X\_{\mathbf{st}}Y\_X\,{}_{s,w}\mathbf{S}\ \_\ Y\,{}_{s,w}\mathbf{S}\_.\ \_6.25\_$

This axiom implies that the only sets consisting entirely of standard sets are of the form $Y\,{}_{s,w}\mathbf{S}$, where $Y\ \_{}_{s,w}\mathbf{S}$.

Axioms for a nonclasscal sets sets of standard size

1. **Saturation for a a nonclasscal sets**

If $A\ \_{}_s\,\mathbf{I}$ is a standard size set then$___X,Y\ _{}_{s,w}A_\ _{}_{s,w}X\,{}_{s,w}Y\ _{}_{s,w}A_\ _\ _X\ _{}_{s,w}A_\ _{}_s X\ _\ _____\ _{}_{s,w}$ ${}_{s,w}A\ \_\ \_.$

$\_6.26\_$

2. **Standard Size Choice for a a nonclasscal sets**

Choice is available in the case where the domain of the choice function is of standard size.Let X be a set of standard size and F a function on X. Then

$\_x\ \_{}_{s,w}X\_\_F\_x\_\ \_\ \_\_\_\_\ \_{}_{s,w}\ \_f\_f\_x\_\ \_{}_{s,w}F\_x\_\_\_\_\ \_6.27\_$

3. **Dependent Choice for a a nonclasscal sets**

Any nonempty partially ordered nonclasscal set without maximal elements includes a nonempty linearly ordered subset (sequence) where any element has its immediate successor.

Conclusion

Though the history of infinitesimals and infinity is long and tortuous,nonstandard analysis, as a canonical formulation of the method of infinitesimals, is only about 60 years old. Hence, definitive answers for many of its methodological issues are yet to be
found.

Acknowledgement

The reviewers for important clarification.

References

[1] J. L. Kelley 1975 (1955) General Topology. Springer. Earlier ed., Van Nostrand. Appendix, "Elementary Set Theory."
[2] A.Tarski, A ber Unerreichbare Kardinalzahlen. Fundamenta Mathematicae 30,68–89 (1938)
[3] M. Kaufmann,Set Theory With a Filter Quantifier, The Journal of Symbolic Logic Vol. 48, No. 2 (Jun., 1983), pp. 263-287, 25 pp.
[4] Foukzon, J. Relevant First-Order Logic $LP_{\#}$ and Curry's Paradox Resolution, Pure and Applied Mathematics Journal. Special Issue: Modern Combinatorial Set Theory and Large Cardinal Properties. Volume 4, Issue 1-1, January 2015, pp. 6-12.
doi: 10.11648/j.pamj.s.2015040101.12
[5] Foukzon, J. (2021). Set Theory INC_#
Based on Intuitionistic Logic with Restricted Modus Ponens Rule (Part. I). Journal of Advances in Mathematics and Computer Science, 36(2), 73-88. https://doi.org/10.9734/jamcs/2021/v36i230339
[6] Foukzon, J. (2021). Set Theory INC_#
Based on Infinitary Intuitionistic Logic with Restricted Modus Ponens Rule (Part.II) Hyper Inductive Definitions. Journal of Advances in Mathematics and Computer Science, 36(4), 90-112.
https://doi.org/10.9734/jamcs/2021/v36i430359
[7] Foukzon, J. (2021). Set Theory INC_#
Based on Innitary Intuitionistic Logic with Restricted Modus Ponens Rule. Hyper Inductive Denitions. Application in Transcendental Number Theory. Generalized Lindemann-Weierstrass Theorem. Journal of Advances in Mathematics and Computer Science, 36(8), 70-119.
https://doi.org/10.9734/jamcs/2021/v36i830394
[8] J. Foukzon,Set Theory NC_#
Based on Bivalent Infinitary Logic with Restricted Modus Ponens Rule. Basic Analysis on External Non-Archimedean Field _c
#.
https://vixra.org/abs/2111.0140
https://papers.ssrn.com/sol3/papers.cfm?abstract_id_3989960
[9] J. Foukzon,The Solution of the Invariant Subspace Problem. Part I. Complex Hilbert Space, 101 Pages Posted: 7 Apr 2022
https://papers.ssrn.com/sol3/papers.cfm?abstract_id_4039068

[10] J. Foukzon,Model P_φ_4 Quantum Field Theory. A Nonstandard Approach Based on Nonstandard Pointwise-Defined Quantum Fields.
https://papers.ssrn.com/sol3/papers.cfm?abstract_id_4163159
[11] E. Nelson, Internal Set Theory, an axiomatic approach to nonstandard analysis, Bull. Am. Math. Soc., 83:6 (1977) 1165-1198.
[12] K. Hrbácek, Axiomatic foundations for nonstandard analysis, Fund. Math. 98, no. 1 (1978) 1–19.
[13] V. Kanovei, M. Reeken, Nonstandard Analysis, axiomatically, Springer Monographs in Mathematics (2004).
[14] A. Robinson,Non-standard analysis. (Reviwed re-edition of the 1st edition of (1966) Princeton: Princeton University Press,1996.
[15] Stroyan, K.D., Luxemburg, W.A.J. Introduction to the theory of infinitesimals. New York: Academic Press (1st ed.), 1976
[16] S. Albeverio, J. E. Fenstad, R. Høegh-Krohn,Nonstandard Methods in Stochastic Analysis and Mathematical Physics (Dover Books on Mathematics) , February 26, 2009 Paperback : 526 pages ISBN-10 : 0486468992,ISBN-13 : 978-0486468990
[17] M. Davis, Applied Nonstandard Analysis (Dover Books on Mathematics) ISBN-13: 978-0486442297 ISBN-10: 0486442292
[18] P. Aczel, Non-well-founded sets, CSLI Publications, Stanford (1988).
[19] B. C. van Fraassen, Formal Semantics and Logic Hardcover,Publisher : Macmillan; First Printing edition (January 1, 1971) Hardcover 256 pages ISBN-10:0024226904 ISBN-13: 978-0024226907

Appendix A. Bivalent Hyper Infinitary first-order logic $2L\_\#$ # with restricted rules of conclusion.Generalized Deduction Theorem.

Hyper infinitary language $L\_\#$
$\#$ are defined according to the length of hyper infinitary conjunctions/disjunctions as well as quantification it allows. In that way, assuming a supply of $\_ \_ \_0$
$\# \_ card\_\_\# \_$ variables to be interpreted as ranging over a nonempty domain, one includes in the inductive definition of formulas an infinitary clause for conjunctions and disjunctions, namely, whenever the hypernaturals indexed hyper infinite sequence $\_A \_ \quad \_\#$ of formulas has length less than _, one can form the hyperfinite conjunction/disjunction of them to produce a formula. Analogously, whenever an hypernaturals indexed sequence of variables has length less than , one can introduce one of the quantifiers _ or _ together with the sequence of variables in front of a formula to produce a new formula. One also stipulates that the length of anywell-formed formula is less than $\_0$
$\#$ itself.

The syntax of bivalent hyper infinitary first-order logics $2L\_\#$
$\#$ consists of a (ordered) set of sorts and a set of function and relation symbols, these latter together with the corresponding type, which is a subset with less than $\_0$
$\# \_ card\_\_\# \_$ many sorts.

Therefore, we assume that our signature may contain relation and function symbols on
$\_ \_ \_0$

many variables, and we suppose there is a supply of _ _ _0
many fresh
variables of each sort. Terms and atomic formulas are defined as usual, and general formulas are defined inductively according to the following rules.
If _,, _,_ : _ _ _ _ (for each _ _ _) are formulas of $L_{\#}$
#, the following are also
formulas:
(i) _ _ _ _,_ _ _ _,
(ii) _ _ _ _,_ _ _ _,
(iii) _ _ ,_ _ ,_ ,_
(iv) _ _ _x_ (also written _**x**_ if **x**_ _ _x_ : _ _ _ _),
(v) _ _ _x_ (also written _**x**_ if **x**_ _ _x_ : _ _ _ _),
(vi) the statement _ _ _ _ holds if and only if for any _ such that _ _ _
the statement holds _ _,
(vii) the statement _ _ _ _ holds if and only if there exist _ such that _ _ _
the statement holds _ _.
Definition 1.[19]. A valuation of a syntactic system is a function that as signs (true) to some of its sentences, and/or (false) to some of its sentences.Precisely, a valuation maps a nonempty subset of the set of sentences into the set _ ,_.
We call a valuation bivalent iff it maps all the sentences into _ ,_.
Definition 2.[19].Let L be a propositional language. L is a classical bivalent propositional
language iff its admissible valuations are the functions v such that for all sentences A,B
of L the following properties hold
(a) v_A_ _ _ ,_
(b) v_ _A_ _ iff v_A_ _
(c) v_A _ B_ _ iff v_A_ _ v_B_ _ .
(d) by definition of the classical implication A _ B the following truth table holds
_1_
_2_
_3_
_4_
v_A_ v_B_ v_A _ B_

Truth table 1.
Remark 1.Note that in the case (4) on a truth table 1
In this case we call implication A _ B a weak implication and abbreviate
A _$_w$$B$ _1_
We call a statement (1) as a weak statement and often abbreviate v_A _ B_ _ $_w$
instead (1). **Definition 3**.[19]. A is a valid (logically valid) sentence (in symbols, _ A) in L iff every admissible valuation of L satisfies A.
The axioms of hyper infinitary first-order logic $_2L_{\#}$
consist of the following schemata:
I. **Logical axiom**

A 1. *A* _ _*B* _ *A*_
A 2. _*A* _ _*B* _ *C*_ _ __*A* _ *B*_ _ _*A* _ *C*___
A 3. __*B* _ _*A*_ _ _*A* _ *B*_
A 4. ___*i*_‗_*A* _ *Ai*___ _ _*A* _ __*i*_ *Ai*_,_ _ _#
A 5. ___*i*_ *Ai*_ _ *Aj*,_ _ _#
A 6. __**x**_*A* _ *B*_ _ _*A* _ _**x***B*__
provided no variable in **x** occurs free in *A*;
A 7. _**x***A*_**x**_ _ *Sf*_*A*_,
where *Sf*_*A*_ is a substitution based on a function *f* from **x** to the terms of the language; in particular:
A 7_. _*xi*_*A*_*xi* ___ _ *A*_**t**_ is a wff of 2*L*_#
and **t** is a term of 2*L*_#
that is free for *xi*
in *A*_*xi* _. Note here that **t** may be identical with *xi*; so that all wffs _*xiA* _ *A*
are axioms by virtue of axiom (7),see [19].
A 8.Gen (Generalization).
_*xiB* follows from *B*.
II.**Restricted rules of conclusion**.
Let _wff be a set of the all closed wffs of *L*_#
.
R1.RMP (Restricted Modus Ponens).
There exist subsets 1,2 _ _wff such that the following rules are satisfied.
From *A* and *A* _ *B*, conclude *B* iff *A* _ 1 and _*A* _ *B*_ _ 2,where 1,2 _ _wff.
In particular for any *A*,*B* _ _wff : *A* _*w* *B* _ 2.
If *A* _ 1 and _*A* _ *B*_ _ 2 we also abbraviate by *A*,*A* _ *B* _*RMP* *B*.
R2.RMT (Restricted Modus Tollens)
There exist subsets 1
_ ,2
_ _ _wff such that the following rules are satisfied.
P _ *Q*,_*Q* _*RMT* _*P* iff *P* _ 1
_ and _*P* _ *Q*_ _ 2
_ ,where 1
_ ,2
_ _ _wff.
Remark 2.Note that RMP and RMT easily prevent any paradoxes of naive Cantor set theory (NC), see [4]-[6].
III.**Additional derived rule of conclusion**.
Particularization rule (RPR)
Remind that canonical unrestricted particularization rule (UPR) reads
UPR: If **t** is free for *x* in *B*_*x*_, then _*x*_*B*_*x*__ _ *B*_**t**_, see [19].
Proof.From _*x*_*B*_*x*__ and the instance _*x*_*B*_*x*__ _ *B*_**t**_ of axiom (A7), we obtain *B*_**t**_ by unrestricted modus ponens rule.Since *x* is free for *x* in *B*_*x*_, a special case of unrestricted particularization rule is:_*xB* _ *B*.
Definition 4.Any formal theory *L* with a hyper infinitary lenguage *L*_#
is defined
when the following conditions are satisfied:
1. A hyper infinite set of symbols is given as the symbols of *L*. A finite or hyperfinite sequence of symbols of *L* is called an expression of *L*.

2. There is a subset of the set of expressions of L called the set of well formed formulas (wffs) of L. There is usually an effective procedure to determine whether a given expression is a wff. 3. There is a set of wfs called the set of axioms of L. Most often, one can
effectively decide whether a given wff is an axiom; in such a case, L is called an axiomatic theory.
4. There is a finite set $R_1, . . . ,R_n$, of relations among wffs, called rules of conclusion. For each R_i, there is a unique positive integer j such that, for every set of j wfs and each wff B, one can effectively decide whether the given j wffs are in the relation R_i to B, and, if so, B is said to follow from or to be a direct consequence of the given wffs by virtue of R_j.
Definition 5.A proof in L is a finite or hyperfinite sequence $B_1, . . . ,B_k$, k _ _# of wffs such that for each i,either B_i is an axiom of L or B_i is a direct consequence of some of the preceding wffs in the sequence by virtue of one of the rules of inference of L.
Definition 6. A theorem of L is a wff B of Y such that B is the last wff of some proof in L. Such a proof is called a proof of B in L.
Definition 7. A wff E is said to be a consequence in L of a set of _ of wffs if and only if there is a finite or hyperfinite sequence $B_1, . . . ,B_k$, k _ _# of wffs such that E is B_k and, for each i,either B_i is an axiom or B_i is in _, or B_i is a direct consequence by some rule of inference of some of the preceding wffs in the sequence. Such a sequence is colled a proof (or deduction) E from _. The members of _ are called the hypotheses or premisses of the proof.
We use _ _ E as an abbreviation for E as a consequence of _.
In order to avoid confusion when dealing with more than one theory, we write _ _$_L$ E, adding the subscript L to indicate the theory in question.
If _ is a finite or hyperfinite set _H_i_1_i_m,m _ _# we write $H_1, . . . ,H_m$ _ E instead of _H_i_1_i_m _ E.
Lemma 1.[19]. _ B _ B for all wffs B.
Theorem 1.(Generalized Deduction Theorem1). If _ is a set of wffs and B and E are wffs, and _,B _ E, then _ _ B _$_s$ E. In pticular, if B _ E then _ B _ E.
Proof. Let $E_1, . . . ,E_n$, n _ _# be a proof of E form _ _ _B_, where E_n is E.
Let us prove, by hyperfinite induction on j, that _ _ B _$_s$ E_j for 1 _ j _ n.
First of all, E_1 must be either in _ or an axiom of L or B itself.
By axiom schema A1, E_1 _$_s$ _B _$_s$ E_1 _ is an axiom. Hence, in the first two cases, by MP, _ _ B _$_s$ E_1 For the third case, when E_1 is B, we have _ B _$_s$ E_1 by Lemma 1, and, therefore, _ _ B _$_s$ E_1. This takes care of the case j _ 1.
Assume now that: _ B _$_s$ E_k for all k _ j, j _ _#. Either E_j is an axiom, or E_j is in _, or E_j is B, or E_j follows by modus ponens from some E_l and E_m where l _ j, m _ j, and E_m has the form E_l _$_s$ E_j. In the first three cases, _ _ B _$_s$ E_j as in the case j _ 1 above. In the last case, we have, by inductive hypothesis, _ _ B _$_s$ E_l and _ _ B _$_s$ _E_l _$_s$ E_j _ But, by axiom schema (A2),
_ B _$_s$ _E_l _$_s$ E_j _ _$_s$ __B _$_s$ E_l _ _$_s$ _B _$_s$ E_j __
Hence, by MP, _ _ _B _$_s$ E_l _ _$_s$ _B _$_s$ E_j _ and, again by MP, _ _ B _$_s$ E_j.
Thus, the proof by hyperfinite induction is complete.
The case j _ n _ _# is the desired result. Notice that, given a deduction of E from _ and B, the proof just given enables us to construct a deduction of B _$_s$ E

from _. Also note that axiom schema A3 was not used in proving thegeneralized deduction theorem.

Remark 3.For the remainder of the chapter, unless something is said to the contrary, we shall omit the subscript L in $\_L$. In addition, we shall use $\_,B$ _ E to stand for _ _ $\_B\_$ _ E. In general, we let $\_,B_1, \ldots ,B_n$ _ E stand for _ _ $\_B_{i\_1\_i\_n}$ _ E.

Remark 4.We shall use the terminology proof, theorem, consequence, axiomatic, etc. and notation _ _ E introduced above.

Proposition 1. Every wff B of K that is an instance of a tautology is a theorem of K, and it may be proved using only axioms A1-A3 and MP.

Proposition 2.If E does not depend upon B in a deduction showing that $\_,B$ _ E, then _ _ E.

Proof.Let $D_1, \ldots ,D_n$ be a deduction of E from _ and B, in which E does not depend upon B. In this deduction, D_n is E. As an inductive hypothesis, let us assume that the proposition is true for all deductions of length less than n _ $\_\#$
If E belongs to _ or is an axiom, then _ _ E. If E is a direct consequence of one or two preceding wffs by Gen or MP, then, since E does not depend upon B, neither do these preceding wfs. By the inductive hypothesis, these preceding wfs are deducible from _ alone. Consequently, so is E .

Theorem 2.(Generalized Deduction Theorem 2).Assume that, in some deduction showing that $\_,B$ _ E, no application of Gen to a wff that depends upon B has as its quantified variable a free variable of B. Then _ _ $B\ \__s E$.

Proof.Let $D_1, \ldots ,D_n$ be a deduction of E from _ and B satisfying the assumption of this theorem. In this deduction, D_n is E. Let us show by hyperfinite induction that _ _ $B\ \__s D_i$ for each i _ n _ $\_\#$. If D_i is an axiom or belongs to _, then _ _ $B\ \__s D_i$, since $D_i\ \__s\ \_B\ \__s D_i\ \_$ is an axiom. If D_i is B, then _ _ $B\ \__s D_i$, since, by Proposition 1, _ $B\ \__s B$. · If there exist j and k less than i such that D_k is _ $D_j\ \__s D_i$, then, by inductive hypothesis, _ _ $B\ \__s D_j$ and _ _ $B\ \__s\ \_D_j\ \__s D_i\ \_$. Now, by axiom A2, _ $B\ \__s\ \_D_j\ \__s D_i\ \_\ \__s\ \_\_B\ \__s D_j\ \_\ \__s\ \_B\ \__s D_i\ \_\_$.Hence, by MP twice, _ _ $B\ \__s D_i$. Finally, suppose that there is some j _ i such that D_i is $\_x_k D_j$. By the inductive hypothesis, _ _ $B\ \__s D_j$, and, by the hypothesis of the theorem, either D_j does not depend upon B or x_k is not a free variable of B. If D_j does not depend upon B, then, by Proposition 2, _ _ D_j and, consequently, by Gen, _ _ $\_x_k D_j$. Thus, _ _ D_i. Now, by axiom A1, _ $D_i\ \__s\ \_B\ \__s D_i\ \_$. So, _ _ $B\ \__s D_i$ by MP. If, on the other hand, x_k is not a free variable of B, then, by axiom A5, _ $\_x_k\ \_B\ \__s D_j\ \_\ \__s\ \_B\ \__s\ \_x_k D_j\ \_$ Since _ _ $B\ \__s D_j$, we have, by Gen,_ _ $\_x_k\ \_B\ \__s D_j\ \_$, and so, by MP,_ _ $B\ \__s\ \_x_k D_j$ that is, _ _ $B\ \__s D_i$. This completes the induction, and our proposition is just the special case i _ n.