A Note on _-stable and _-inverse Gaussian Laws

Abstract
In this article we obtain the _rst passage time distribution of _-stable L_evy processes. We derive the moment estimators of the parameters of _-inverse Gaussian laws and also their asymptotic distribution.

Key Words: asymptotic normality, Brownian motion, estimator, _rst passage time, inverse Gaussian, Laplace transform, stable.

1 Introduction

_-stable laws are in_nitely divisible and hence one can de_ne corresponding _-stable L_evy processes (_SLP). The range of _ is 0 < _ _ 2 and for _ = 2 the _-stable law is Gaussian/ normal law and the corresponding _SLP is the Brownian motion process (BMP). It is known that $\frac{1}{2}$-stable law is the _rst passage time (FPT) distribution of BMP with zero drift (Feller, 1971, p.174) and for a BMP with positive drift, the FPT distribution is inverse Gaussian (IG), Johnson and Kotz (1970, p.137). IG laws were generalized to _-IG (_IG) laws in Pillai and Satheesh (1992).

Here, in section 2, we obtain the FPT distribution of _SLP. Since the density of _IG laws is not in closed form we derive the moment estimators of its parameters and obtain their asymptotic distribution in section 3. Another possible approach based on the p.d.f. of Gamma is also sketched.

2 FPT distribution of _SLP

We need the following result from Eaton et al. (1971) to de_ne the _SLP. They call it extreme stable since the parameter _ in the stable model is set as _ = 1. They have taken the location parameter also as zero. However, here we refer to them as _-stable laws.

Theorem 2.1. The function M(s) = expf□b(1□_)s_g; 0 _ Re(s) < 1; 0 < _ _ 2; _ 6= 1; b > 0 are moment generating functions (MGF) of _-stable laws.

De_nition 2.2. L_evy processes fX(t); t _ 0g are _SLP, if the distribution of X(1) is _-stable with MGF M(s) = expfb(_ □ 1)s_g.

FPT distributions of processes are important as they give the distribution of the time taken for the process to reach/ cross a barrier. If _ > 0 is the barrier, then the random variable (r.v.) T(_) = T = infft > 0 : X(t) _ _g denote the FPT of X(t). Here t > 0, since X(0) = 0 for a L_evy process. Since the location parameter is zero for the _-stable laws considered here, the _SLP has zero drift. Further, for 1 < _ _ 2, it has _nite mean and hence martingale based arguments on X(t) are justi_ed. We now derive the FPT distribution of _SLP using standard arguments based on optional sampling theorem (see, Karlin and Taylor, 1975) applied to the martingale of fX(t)g in proposition 2.3. Other relevant literature are: Novikov (1981, 2009), Alili and Kyprianov (2005), Schilling and Partzsch (2014), Vellaisamy and Kumar (2017).

Proposition 2.3. For the _SLP fX(v); v _ 0g, W(v) = expfsX(v) □ _vg,

s > 0 a constant, is a martingale, where _ = b(_ □ 1)s_.
Proof. Since, E
□
esX(t)
_
= e_t, E(jW(v)j) = E(W(v)) = e□_vE
□
esX(v)
_
= 1 <
1. Since L_evy processes have stationary and independent increments, for
u _ v, X(v) □ X(u) is independent of Fu, the _ltration up to time u. Now,
E (W(v)=Fu) = E (expfsX(v) □ _v=Fug)
= e□_vE
□
es[X(v)□X(u)]+sX(u)=Fu
_
= e□_vE
□
es[X(v)□X(u)]=Fu
_
E
□
esX(u)=Fu
_
= e□_vE
□
esX(v□u)=Fu
_
esX(u)
= e□_ve_(v□u)esX(u) = esX(u)□_u = W(u);

and that completes the proof.
Theorem 2.4. The FPT distribution of _SLP for 1 < _ _ 2, is 1
_-stable.
Proof. Let the r.v. T(_) = T denote the FPT for the _SLP fX(t); t _ 0g
to reach or cross _ > 0. We know that for fX(t)g, W(t) = expfsX(t) □ _tg
is a martingale, where _ = b(_ □ 1)s_. For a martingale fW(t)g and for the
FPT T (which is a stopping time), EfW(0)g = EfW(T ^ t)g. As X(0) = 0,
W(0) = 1 and hence EfW(T ^ t)g = 1. That is,
E [expfsX(T ^ t) □ _(T ^ t)g] = 1; (1)
Note that _ = b(_ □ 1)s_ > 0 for 1 < _ _ 2 and so 0 _ W(T ^ t) _ es_:
Now assuming PfT < 1g = 1 (we will justify this at the end of the proof)
we may pass to the limit as t ! 1under the expectation in (1) by the optional
sampling theorem, yielding;
1 = lim
t!1

E [expfsX(T ^ t) □ _(T ^ t)g] = es_E
_
e□_T _
:
Thus; E
_
e□_T _
= e□s_:
Since _ = b(_ □ 1)s_ =) s =
n
_
b(_□1)
o1=_
, we get the LT of the FPT as,
E
_
e□_T _
= e
_
□_
[b(_□1)]1=_
_
_1=_
;
which is that of 1
_-stable law (Feller, 1971, p.448).
Finally, since PfT < 1g = lim_#0 E
_
e□_T
_
= 1; T has a proper distribu-
tion, justifying our assumption PfT < 1g = 1.
Remark 2.1. E
_
e□_T
_
= e□__1=_ is the LT of a probability distribution only
when _1=_ has completely monotone derivative. That is, if 0 < 1=_ < 1 =)
_ > 1 (Feller 1971, p.448). Also, here we need _ > 0. These are the reasons
for restricting the range of _ to 1 < _ _ 2 in the above theorem. When _ = 1
the distribution is degenerate.
Remark 2.2. One may note that when _ = 2, the _SLP is BMP and the LT
of T is E
_
e□_T
_
= e
h
□p_
b
i

_1=2
which is that of 1
2 -stable, as is known.

3 Estimation of Parameters of _IG

By Pillai and Satheesh (1992), a r.v. X _ _IG(_; _;m) (_IG), if its LT is;
L(s) = expf
m
_
[1 □ (1 +
2_2
m
s)_]g; s _ 0; 0 < _ < 1; _; m > 0:
The probability density function (p.d.f.) of X _ _IG(_; _;m) is;
f(x) =
1
c
exp
_
m
_
_
1 □
x
2_
__
p_
_x
c
_
; x > 0; where c = 2m
1
_
□1_2□ 1
_;
and p_(x) =
1
_x
X1
k=1
(□1)k□1 □(_k + 1)
k!
sin(k__)(x□_)k; x > 0;
is the p.d.f. of _-stable law. Since the p.d.f. of _IG law is complex, estimation based on p.d.f.s is not easy and so we adapt that in Padgett and Wei (1979).
From the logarithm of the LT of _IG we get its rth cumulant as:
_r =
_

@r log L(s)
@sr
_
s=0
=
□m
_
_(_□1):::(_□r+1)
_
□2_2
m
_r
=
□m
_
(_; r)
_
□2_2
m
_r
:
Using the relations (Rao, 1973, p.101) connecting _j and _j the central moments, (for direct computation see, Pillai and Satheesh (1992)) we have,
_1 = _1 =
m
_
_
2_2
m
= 2__; (2)
_2 = _2 =
□m
_
_(_ □ 1)
_
□2_2
m
_2
= 4_(1 □ _)
_3
m
and (3)
_3 = _3 =
□m
_
_(_ □ 1)(_ □ 2)
_

□2_2
m
_3
= 8_(_ □ 1)(_ □ 2)
_5
m2 : (4)
From (2) and (3) =) 2_1(1 □ _)
_2
m
= _2; (5)
from (3) and (4) =) 2_2(2 □ _)
_2
m
= _3; and (6)
from (5) and (6) =)
_1(1 □ _)
_2(2 □ _)
=
_2
_3
: (7)

Solving; from (7); _ =
2_22
□ _1_3
_22
□ _1_3
= 1 +
_22
_22
□ _1_3
;
from (2); _ =
_1
2_
=
_1
2
_
_22
□ _1_3
2_22
□ _1_3
_
and
from (5); m =

2_1
_2
(1 □ _)_2 =
_31
_2
2
_1_3 □ _22
(2_22
□ _1_3)2 :

Let x1; :::; xn be a simple random sample of size n from X _ _IG(_; _;m) and a, b, c respectively denote the mean, variance and third central moment from the sample. Now the moment estimators of the parameters _; _ and m are;

^_ =
2b2 □ ac
b2 □ ac
= 1 +
b2
b2 □ ac
;
^_ =
a
2_
=
a
2
_
b2 □ ac
2b2 □ ac
_
and
^m =
2a
b
(1 □ _)_2 =
a3b
2
ac □ b2
(2b2 □ ac)2 :

Remark 3.1. From remark 5 in Pillai and Satheesh (1992), Gamma(_, 2/_) law is the mixture of X _ _IG(_; _; __2) with _ _ Exp(_). The corresponding stochastic representation is Y = XE. This opens up the possibility of transforming the observations on _IG r.v. X to the corresponding gamma r.v. Y , estimating the parameters of the gamma and in turn that of the _IG. Comparing these estimates with the moment estimates obtained above for their e_ciency and closeness is worth investigating. This demands simulation studies and will be reported elsewhere.

3.1 Asymptotic distribution of the estimators

If X _ _IG(_; _;m), then the vector $(X;X_2;X_3)_0$ has mean _ = $(\_{1}; \_{2}; \_{3})_0$
and covariance matrix
_ =
2
64
$\_{2}$ □ $\_{2}$
1 $\_{3}$ □ $\_{1}\_{2}$ $\_{4}$ □ $\_{1}\_{3}$
$\_{4}$ □ $\_{2}$
2 $\_{5}$ □ $\_{2}\_{3}$
$\_{6}$ □ $\_{2}$
3
3
7 5,
where $\_{k}$ = E
□
X_k
_
; k = 1; 2; :::; 6. De_ne _X
$_j$ = 1
n
P_n
i=1 X_j
$_i$; j = 1; 2; 3 and

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put _X
= (_X
1; _X
2; _X
3)0. Then, by the multivariate CLT (Rao, 1973, p.128),
p
n(_X
□ _) d□
! Z_1 _ $N_3(0; \_)$:
Using the relations between $\_{j}$ and $\_{j}$ (Rao, 1973, p.101), de_ne the following functions g_i, so that we get g_i(_X
); i = 1; 2; 3, the corresponding moment
estimators.
$g_1(\_) = g_1(\_{1}; \_{2}; \_{3})$ = _ = 1 +
$(\_{2}$ □ $\_{2}$
$1)^2$
$(\_{2}$ □ $\_{2}$
$1)^2$ □ $\_{1}(\_{3}$ □ $3\_{1}\_{2} + 2\_{2}$
1)
;
$g_2(\_) = g_2(\_{1}; \_{2}; \_{3})$ = _ =
$\_{1}$

2_
; and
$g_3(\_) = g_3(\_1; \_2; \_3) = m = 2$
$\_^1$
$\_^2$ □ $\_^2$
1
(1 □ _)$\_^2$:
Since g_i's are totally di_erentiable, we get the asymptotic joint distribution of the moment estimators invoking the result (iii) in Rao (1973, p.388) as,
p
n
□
g_1(_X
) □ g_1(_); g_2(_X
) □ g_2(_); g_3(_X
) □ g_3(_)
_ d□
! Z_2 _ $N_3(0;G\_G_0)$;
where G = @g_i(_)
@_j
.
Pillai and Satheesh (1992) showed that _IG laws are self-decomposable. This motivated Abraham and Balakrishna (1999) to develop AR(1) models with IG marginals. Sri Ranganath and Balakrishna (2019) discussed Bayesian analysis of IG stochastic conditional duration models. Vellaisamy and Kumar (2017) derived the FPT distribution of IG process. These suggest the possibility of discussing _IG laws in other modelling contexts.

Summary In this paper the FPT distribution of _SLP, for 1 < _ _ 2, is obtained as 1
_-stable law. Moment estimators of the parameters of _IG law are derived and they are shown to be jointly asymptotically normal.

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