A Note on α-stable and α-inverse Gaussian Laws

Abstract

In this note we obtain the first passage time distribution of α-stable Lévy processes. We derive the moment estimators of the parameters of α-inverse Gaussian laws and also their asymptotic distribution. [1]

[1]Key Words: asymptotic normality, Brownian motion, estimator, first passage time, inverse Gaussian, Laplace transform, stable

1 Introduction

α-stable laws are infinitely divisible and hence one can define corresponding α-stable Lévy processes (αSLP). The range of α is $0 < \alpha \leq 2$ and for $\alpha = 2$ the α-stable law is Gaussian/ normal law and the corresponding αSLP is the Brownian motion process (BMP). It is known that $\frac{1}{2}$-stable law is the first passage time (FPT) distribution of BMP with zero drift (Feller, 1971, p.174) and for a BMP with positive drift, the FPT distribution is inverse Gaussian (IG), Johnson and Kotz (1970, p.137). IG laws were generalized to α-IG (αIG) laws in Pillai and Satheesh (1992).

Here, in section 2, we obtain the FPT distribution of αSLP for $1 < \alpha \leq 2$. Since the density of αIG laws is not in closed form we derive the moment estimators of its parameters and obtain their asymptotic distribution in section 3. Another possible approach based on the *p.d.f.* of Gamma is also sketched.

2 FPT distribution of αSLP for $1 < \alpha \leq 2$

We need the following result from Eaton *et al.* (1971) to define the αSLP for $1 < \alpha \leq 2$. They call it extreme stable since the parameter β in the stable model is set as $\beta = 1$. They have taken the location parameter also as zero. However, here we refer to them as α-stable laws.

Theorem 2.1. *The function* $M(s) = \exp\{-b(1-\alpha)s^{\alpha}\}$; $0 \leq Re(s) < \infty$; $0 < \alpha \leq 2, \alpha \neq 1, b > 0$ *are moment generating functions (MGF) of* α*-stable laws.*

Definition 2.2. *Lévy processes* $\{X(t); t \geq 0\}$ *are* α*SLP, if the distribution of* $X(1)$ *is* α*-stable with MGF* $M(s) = \exp\{b(\alpha - 1)s^{\alpha}\}$.

FPT distributions of processes are important as they give the distribution of the time taken for the process to reach/ cross a barrier. If $\lambda > 0$ is the barrier, then the random variable (*r.v.*) $T(\lambda) = T = \inf\{t > 0 : X(t) \geq \lambda\}$ denote the FPT of $X(t)$. Here $t > 0$, since $X(0) = 0$ for a Lévy process. Since the location parameter is zero for the α-stable laws considered here, the αSLP has zero drift. Further, for $1 < \alpha \leq 2$, it has finite mean and hence martingale based arguments on $X(t)$ are justified. We now derive the FPT distribution of αSLP using standard arguments based on optional sampling theorem applied to the following martingale of $\{X(t)\}$.

Proposition 2.3. *For the* α*SLP* $\{X(v), v \geq 0\}$, $W(v) = \exp\{sX(v) - \theta v\}$, $s > 0$ *a constant, is a martingale, where* $\theta = b(\alpha - 1)s^{\alpha}$.

Proof. Since, $E\left(e^{sX(t)}\right) = e^{\theta t}$, $E(|W(v)|) = E(W(v)) = e^{-\theta v} E\left(e^{sX(v)}\right) = 1 < \infty$. Since Lévy processes have stationary and independent increments, for $u \leq v$, $X(v) - X(u)$ is independent of $\mathcal{F}_u$, the filtration up to time u. Now,

$$\begin{aligned}
E\left(W(v)/\mathcal{F}_u\right) &= E\left(\exp\{sX(v) - \theta v/\mathcal{F}_u\}\right) \\
&= e^{-\theta v} E\left(e^{s[X(v)-X(u)]+sX(u)}/\mathcal{F}_u\right) \\
&= e^{-\theta v} E\left(e^{s[X(v)-X(u)]}/\mathcal{F}_u\right) E\left(e^{sX(u)}/\mathcal{F}_u\right) \\
&= e^{-\theta v} E\left(e^{sX(v-u)}/\mathcal{F}_u\right) e^{sX(u)} \\
&= e^{-\theta v} e^{\theta(v-u)} e^{sX(u)} = e^{sX(u)-\theta u} = W(u),
\end{aligned}$$

and that completes the proof. □

Theorem 2.4. *The FPT distribution of αSLP for $1 < \alpha \leq 2$, is $\frac{1}{\alpha}$-stable.*

Proof. Let the *r.v.* $T(\lambda) = T$ denote the FPT for the αSLP $\{X(t), t \geq 0\}$ to reach or cross $\lambda > 0$. We know that for $\{X(t)\}$, $W(t) = \exp\{sX(t) - \theta t\}$ is a martingale, where $\theta = b(\alpha - 1)s^{\alpha}$. For a martingale $\{W(t)\}$ and for the FPT T (which is a stopping time), $E\{W(0)\} = E\{W(T \wedge t)\}$. As $X(0) = 0$, $W(0) = 1$ and hence $E\{W(T \wedge t)\} = 1$. That is,

$$E\left[\exp\{sX(T \wedge t) - \theta(T \wedge t)\}\right] = 1, \tag{1}$$

Note that $\theta = b(\alpha - 1)s^{\alpha} \geq 0$, and so $0 \leq W(T \wedge t) \leq e^{s\lambda}$.

Now assuming $P\{T < \infty\} = 1$ (we will justify this at the end of the proof) we may pass to the limit as $t \to \infty$ under the expectation in (1) by the optional sampling theorem, yielding;

$$1 = \lim_{t\to\infty} E\left[\exp\{sX(T \wedge t) - \theta(T \wedge t)\}\right] = e^{s\lambda} E\left[e^{-\theta T}\right].$$

$$Thus,\ E\left[e^{-\theta T}\right] = e^{-s\lambda}.$$

Since $\theta = b(\alpha - 1)s^{\alpha} \implies s = \left\{\frac{\theta}{b(\alpha-1)}\right\}^{1/\alpha}$, we get the LT of the FPT as,

$$E\left[e^{-\theta T}\right] = e^{\left[\frac{-\lambda}{[b(\alpha-1)]^{1/\alpha}}\right]\theta^{1/\alpha}},$$

which is that of $\frac{1}{\alpha}$-stable law, Feller, 1971, p.448.

Finally, since $P\{T < \infty\} = \lim_{\theta\downarrow 0} E\left[e^{-\theta T}\right] = 1$, T has a proper distribution, justifying our assumption $P\{T < \infty\} = 1$. $\square$

Remark 2.1. *One may note that when $\alpha = 2$, the αSLP is BMP and the LT of T is $E\left[e^{-\theta T}\right] = e^{\left[\frac{-\lambda}{\sqrt{b}}\right]\theta^{1/2}}$ which is that of $\frac{1}{2}$-stable, as is known.*

3 Estimation of Parameters of αIG

By Pillai and Satheesh (1992), a *r.v.* $X \sim \alpha IG(\alpha, \mu, m)$ (αIG), if its LT is;

$$L(s) = \exp\{\frac{m}{\mu}[1 - (1 + \frac{2\mu^2}{m}s)^{\alpha}]\};\ s \geq 0,\ 0 < \alpha < 1,\ \mu,\ m > 0.$$

The probability density function (*p.d.f.*) of $X \sim \alpha IG(\alpha, \mu, m)$ is;

$$f(x) = \frac{1}{c}\exp\left(\frac{m}{\mu}\left(1-\frac{x}{2\mu}\right)\right)p_\alpha\left(\frac{x}{c}\right),\ x>0,\ where\ c = 2m^{\frac{1}{\alpha}-1}\mu^{2-\frac{1}{\alpha}},$$

$$and\ p_\alpha(x) = \frac{1}{\pi x}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\Gamma(\alpha k+1)}{k!}\sin(k\pi\alpha)(x^{-\alpha})^k,\ x>0,$$

is the *p.d.f.* of α-stable law. Since the *p.d.f.* of αIG law is complex, estimation based on *p.d.f.*s is not easy and so we adapt that in Padgett and Wei (1979).

From the logarithm of the LT of αIG we get its r^{th} cumulant as:

$$\kappa_r = \left[\frac{\partial^r \log L(s)}{\partial s^r}\right]_{s=0} = \frac{-m}{\mu}\alpha(\alpha-1)...(\alpha-r+1)\left[\frac{-2\mu^2}{m}\right]^r = \frac{-m}{\mu}(\alpha,r)\left[\frac{-2\mu^2}{m}\right]^r.$$

Using the relations (Rao, 1973, p.101) connecting κ_j and μ_j the central moments, (for direct computation see, Pillai and Satheesh (1992)) we have,

$$\mu_1 = \kappa_1 = \frac{m}{\mu}\alpha\frac{2\mu^2}{m} = 2\alpha\mu, \tag{2}$$

$$\mu_2 = \kappa_2 = \frac{-m}{\mu}\alpha(\alpha-1)\left(\frac{-2\mu^2}{m}\right)^2 = 4\alpha(1-\alpha)\frac{\mu^3}{m}\ and \tag{3}$$

$$\mu_3 = \kappa_3 = \frac{-m}{\mu}\alpha(\alpha-1)(\alpha-2)\left(\frac{-2\mu^2}{m}\right)^3 = 8\alpha(\alpha-1)(\alpha-2)\frac{\mu^5}{m^2}. \tag{4}$$

$$From\ (2)\ and\ (3) \implies 2\mu_1(1-\alpha)\frac{\mu^2}{m} = \mu_2, \tag{5}$$

$$from\ (3)\ and\ (4) \implies 2\mu_2(2-\alpha)\frac{\mu^2}{m} = \mu_3,\ and \tag{6}$$

$$from\ (5)\ and\ (6) \implies \frac{\mu_1(1-\alpha)}{\mu_2(2-\alpha)} = \frac{\mu_2}{\mu_3}. \tag{7}$$

$$Solving,\ from\ (7),\ \alpha = \frac{2\mu_2^2-\mu_1\mu_3}{\mu_2^2-\mu_1\mu_3} = 1+\frac{\mu_2^2}{\mu_2^2-\mu_1\mu_3},$$

$$from\ (2),\ \mu = \frac{\mu_1}{2\alpha} = \frac{\mu_1}{2}\left(\frac{\mu_2^2-\mu_1\mu_3}{2\mu_2^2-\mu_1\mu_3}\right)\ and$$

$$from\ (5),\ m = \frac{2\mu_1}{\mu_2}(1-\alpha)\mu^2 = \frac{\mu_1^3\mu_2}{2}\frac{\mu_1\mu_3-\mu_2^2}{(2\mu_2^2-\mu_1\mu_3)^2}.$$

Let $x_1, ..., x_n$ be a simple random sample of size n from $X \sim \alpha IG(\alpha, \mu, m)$ and a, b, c respectively denote the mean, variance and third central moment from the sample. Now the moment estimators of the parameters $\alpha,\ \mu$ *and* m are;

$$\hat{\alpha} = \frac{2b^2 - ac}{b^2 - ac} = 1 + \frac{b^2}{b^2 - ac},$$
$$\hat{\mu} = \frac{a}{2\alpha} = \frac{a}{2}\left(\frac{b^2 - ac}{2b^2 - ac}\right) \textit{ and}$$
$$\hat{m} = \frac{2a}{b}(1-\alpha)\mu^2 = \frac{a^3 b}{2}\frac{ac - b^2}{(2b^2 - ac)^2}.$$

Remark 3.1. *From remark 5 in Pillai and Satheesh (1992), Gamma(α, 2/θ) law is the mixture of $X \sim \alpha IG(\alpha, \mu, \theta\mu^2)$ with $\mu \sim Exp(\theta)$. The corresponding stochastic representation is $Y = XE$. This opens up the possibility of transforming the observations on αIG r.v. X to the corresponding gamma r.v. Y, estimating the parameters of the gamma and in turn that of the αIG. Comparing these estimates with the moment estimates obtained above for their efficiency and closeness is worth investigating. This demands simulation studies and will be reported elsewhere.*

3.1 Asymptotic distribution of the estimators

If $X \sim \alpha IG(\alpha, \mu, m)$, then the vector $(X, X^2, X^3)'$ has mean $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)'$ and covariance matrix

$$\Sigma = \begin{bmatrix} \alpha_2 - \alpha_1^2 & \alpha_3 - \alpha_1\alpha_2 & \alpha_4 - \alpha_1\alpha_3 \\ & \alpha_4 - \alpha_2^2 & \alpha_5 - \alpha_2\alpha_3 \\ & & \alpha_6 - \alpha_3^2 \end{bmatrix},$$

where $\alpha_k = E\left(X^k\right),\ k = 1, 2, ..., 6$. Define $\bar{X}^j = \frac{1}{n}\sum_{i=1}^n x_i^j,\ j = 1, 2, 3$ and put $\bar{X} = (\bar{X}^1, \bar{X}^2, \bar{X}^3)'$. Then, by the multivariate CLT (Rao, 1973, p.128),

$$\sqrt{n}(\bar{X} - \underline{\alpha}) \xrightarrow{\text{d}} Z_1 \sim N_3(0, \Sigma).$$

Using the relations between α_j and μ_j (Rao, 1973, p.101), define the following functions g_i, so that we get $g_i(\bar{X}),\ i = 1, 2, 3$, the corresponding moment estimators.

$$
\begin{aligned}
g_1(\underline{\alpha}) &= g_1(\alpha_1, \alpha_2, \alpha_3) = \alpha = 1 + \frac{(\alpha_2 - \alpha_1^2)^2}{(\alpha_2 - \alpha_1^2)^2 - \alpha_1(\alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^2)}, \\
g_2(\underline{\alpha}) &= g_2(\alpha_1, \alpha_2, \alpha_3) = \mu = \frac{\alpha_1}{2\alpha}, \ \textit{and} \\
g_3(\underline{\alpha}) &= g_3(\alpha_1, \alpha_2, \alpha_3) = m = 2\frac{\alpha_1}{\alpha_2 - \alpha_1^2}(1-\alpha)\mu^2.
\end{aligned}
$$

Since g_i's are totally differentiable, we get the asymptotic joint distribution of the moment estimators invoking the result (iii) in Rao (1973, p.388) as,

$$
\sqrt{n}\left(g_1(\bar{X}) - g_1(\underline{\alpha}),\ g_2(\bar{X}) - g_2(\underline{\alpha}),\ g_3(\bar{X}) - g_3(\underline{\alpha})\right) \xrightarrow{\text{d}} Z_2 \sim N_3(\underline{0}, G\Sigma G'),
$$

where $G = \frac{\partial g_i(\underline{\alpha})}{\partial \alpha_j}$.

References

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