UNDER PEER REVIEW

Short Research Article

Probability risk model of claim amount affected by threshold

Abstract

In this paper, we consider a risk model of claim amount affected by threshold. The comparision between the claim interval and the threshold will affect the distribution of claims. The hypothesis of the model is presented and then we derive the roots of the generalised Lundberg equation, the Laplace Transform of the expected discounted penalty function. Besides, the Gerber-Shiu penalty function is given when the initial surplus is zero and when it satifies some defective renewal equations. Some explicit expressions about the ruin probability are given too.

Keywords: Gerber-Shiu penalty function; Laplace Transform; defective renewal equation; ruin probability

1 Introduction

In the actuarial literature, the classial compound Poisson risk model and the risk model based on the renewal or the Sparre Andersen risk model have been extensively investigated. And it is explicitly assumed that the interarrival times between two successive claims and the claim amounts are independent. However, this assumptions is inappropriate in the real world. To avoid this, some papers started to study the dependent risk models. Such as, Nyrhinen(1999)[1], Muller and Pflug(2001)[2], Yuen and Guo(2001)[3]. M. Boudreault et al.(2006)[4] studied the dependence structure among the interclaim time and the subsequent size.Several renewal risk models with different interclaim times have been studied by many authors.H. Cossette et al(2010)[5] and Stathis et al.(2012)[6] considered an extension to the renewal process by introducing a dependence structure between the claim sizes and interclaim times through a Farlie-Gumbel-Morgenstern copula. Zhang and Liu(2020)[7] considered a discrete-time risk model with time-dependent claims and impullsive dividend payments. And some other distributions are also applied to the risk model, see Guan and Hu(2021)[8] and Xu and Wang(2011)[9].

In this paper, we consider the case in which the distribution of the claim size is controlled by a random varible. If the claim arrival times exceeds the random level, the claim size will follow one type of distribution. If not, it will follow another type of distribution.

The paper is organsized as follows. In section 2, we introduce the risk model and some basic assumptions. We analyse the generalised Lundberg equation and its roots in section 3. In section 4,the Laplace Transform (LT) of the Gerber-Shiu expected discount penalty function is given. And then we analyse the Gerber-Shiu penalty function when u=0 in Section 5. In section 6, the defective renewal function is given. Finally, explicit expressions and numerical examples are given in Section 7.

2 The model

In this section, we consider the surplus process $\{U(t), t \geq 0\}$ defined as follows

$$U(t) = u + ct - S(t),$$

where $u = U(0) \geq 0$ is the initial surplus and c is the premium rate which is assumed to be a positive constant. $S(t), t \geq 0$ is the total claim amount process defined by $S(t) = \sum_{i=1}^{N(t)} X_i$ and $\sum_a^b = 0$ if $a > b$. The claim number process $\{N(t), t \geq 0\}$ is a Possion process defined via a sequence of i.i.d. interclaim times $\{W_i\}_{i=1}^{\infty}$. For convenience, we denote the claim arrival times $T_j, j \in N^+$ by $T_j = W_1 + \ldots + W_j$. In this paper, we consider that the r.v. W has an exponentially distribution with expectation $1/\lambda, \lambda > 0$ with p.d.f. given by

$$f_W(t) = \lambda e^{-\lambda t}, t \geq 0.$$

The random variable(r.v.) X_i represents the size of the ith claim. We assume that $M_i, i = 1, 2, \ldots$ a sequence of i.i.d. non-negative random variables distributed as M with Erlang(2) distribution with expectation $2/\beta, \beta > 0$ with p.d.f. given by

$$f(t) = \beta^2 t e^{-\beta t}, t \geq 0.$$

Then the claim sizes are determined as follows: If T_i is smaller than M_i, then the following claim size X_i has density function $f_1(x)$, otherwise its density function is $f_2(x)$. Here $M_i, i = 1, 2, \ldots$ are independent of T_i and X_i. From above notations, we get that

$$P(M \leq T) = 1 - e^{-\beta t} - \frac{1}{\beta} f(t)$$

$$P(M > T) = e^{-\beta t} + \frac{1}{\beta} f(t)$$

The risk model with dependence structure can be seen as a more realistic model than the classical compound Poisson risk model, the former approximates the behaviour of the aggregate claim process in a natural cause context. Besides, suppose W_j is the waiting time between the (j-1)th and jth causes. Such a cause has two possible intensities, say $I_j = 1$(usual), 2(severe). It derives

$$Pr(I_j = 1 | W_j = w) = e^{-\beta w} = 1 - Pr(I_j = 2 | W_j = w)$$

and then $Pr(X_j | I_j = i) = F_i(x)$ for i=1,2.

We let $\tau = inf_{t \geq 0}\{t, U_t < 0\}$ be the time of ruin with $\tau = \infty$ if $U_t \geq 0$(i.e. ruin does not occur). The deficit at ruin is denoted by $|U_\tau|$ and the surplus just prior to ruin is $U_{\tau-}$. The Gerber-Shiu discounted penalty function $m_\delta(u)$ is defined as

$$m_\delta(u) = E[e^{-\delta\tau} w(U_{\tau-}, |U_\tau|) 1_{\tau<\infty} | U_0 = u],$$

where $\delta > 0, w : R^+ \times R^+ \to R^+$ is the penalty function. Especially, a special case of the Gerber-Shiu discounted penalty function is the infinite-time ruin probability $\psi(u) = Pr(\tau < \infty)$. To ensure that ruin does not almost surely occur, the premium rate c is such that

$$E[cW_j - X_j] > 0, j = 1, 2, \ldots \tag{1}$$

providing a positive safety loading.

3 Lundberg's generalised equation

In this section, we derive a Lundberg's generalised equation. We consider the discrete-time process embedded in the continuous-time surplus process $\{U(t); t \geq 0\}$. Define the discrete-time process by $U_0 = u$ and for $k = 1, 2, \dots,$

$$U_k = u + \sum_{i=1}^{k} (cW_i - X_i),$$

to be the surplus immediately after the kth claim. We seek a number such that the process $\{e^{-\delta \sum_{i=1}^{k} W_i + sU_k}, \quad k = 0, 1, 2, \dots\}$ for $s > 0$ is a martingale if and only if

$$E[e^{-\delta W} e^{s(cW-X)}] = E[e^{(cs-\delta)W} e^{-sX}] = 1, \tag{2}$$

which is called the *generalised Lundberg equation* associated with our risk model. From the definition in scetion 2, the left-hand side of Equation (2) can be written as

$$\begin{aligned} E[e^{-\delta W} e^{s(cW-X)}] &= \int_0^\infty \int_0^\infty e^{-(\delta - cs)t} K(t) P(M > T) f_1(x) e^{-sx} dx dt \\ &+ \int_0^\infty \int_0^\infty e^{-(\delta - cs)t} K(t) P(M \leq T) f_2(x) e^{-sx} dx dt \\ &= \int_0^\infty \int_0^\infty e^{-(\delta - cs)t} \lambda e^{-\lambda t} \left[e^{-\beta t} + \frac{1}{\beta} \beta^2 t e^{-\beta t} \right] f_1(x) e^{-sx} dx dt \\ &+ \int_0^\infty \int_0^\infty e^{-(\delta - cs)t} \lambda e^{-\lambda t} \left[1 - e^{-\beta t} - \frac{1}{\beta} \beta^2 t e^{-\beta t} \right] f_2(x) e^{-sx} dx dt \\ &= \frac{\lambda}{c} \left[\frac{(\lambda + \beta + \delta - cs + \beta) \hat{f}_1(s)}{c(\frac{\lambda + \beta + \delta}{c} - s)^2} + \frac{\beta^2 \hat{f}_2(s)}{c^2 (\frac{\lambda + \beta + \delta}{c} - s)^2 (\frac{\lambda + \delta}{c} - s)} \right]. \end{aligned} \tag{3}$$

Then, Lundberg's generalised equation in (2) reduces to

$$\frac{\lambda}{c} \frac{\left(\frac{\lambda + \delta}{c} - s\right) \left[\frac{1}{\beta}(\lambda + \beta + \delta - cs) + 1\right] \hat{f}_1(s) + \frac{\beta}{c} \hat{f}_2(s)}{\frac{c}{\beta} \left(\frac{\lambda + \delta}{c} - s\right) \left(\frac{\lambda + \beta + \delta}{c} - s\right)^2} = 1. \tag{4}$$

We use Rouche´'s theorem to show the numbers of roots of the generalized Lundberg equation in the following proposition.

PROPOSITION 1. For $\delta > 0$, Lundberg's generalised equation in (4) has exactly 3 roots, say $\rho_1(\delta)$, $\rho_2(\delta)$, $\rho_3(\delta)$, with $Re(\rho_i(\delta)) > 0$, $i = 1, 2, 3, 3$.

Proof. The generalised Lundberg Equation (4) also becomes

$$\begin{aligned} &\lambda(\lambda + \delta - cs)(\lambda + 2\beta + \delta - cs) \hat{f}_1(s) + \lambda \beta^2 \hat{f}_2(s) \\ &= (\lambda + \delta - cs)(\lambda + \beta + \delta - cs)^2, \end{aligned} \tag{5}$$

it can be seen that the above equation has exactly 3 roots with positive real parts. We denote by C_r the contour containing the imaginary axis running from -ir to ir and a semicircle with radius r running clockwise fron -ir to ir, that is, $C_r = \{s \in C : |s| = r, Re(s) \geq 0, r > 0\}$. We apply Rouche's theorem on the closed contour C to prove the result.

(1) For $Re(s) > 0$, we have $|\lambda + \beta - cs| \to \infty$, $\quad |\delta + \lambda + \beta - cs| \to \infty$ as $r \to \infty$, and thus

$$\Big| \left[\frac{\lambda}{(\lambda+\beta+\delta-cs)} + \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right] \hat{f}_1(s) + \left[\frac{\lambda}{(\lambda+\delta-cs)} - \frac{\lambda}{(\lambda+\beta+\delta-cs)} - \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right] \hat{f}_2(s) \Big|$$
$$\leq \left| \frac{\lambda}{(\lambda+\beta+\delta-cs)} + \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right| |\hat{f}_1(s)| + \left| \frac{\lambda}{(\lambda+\delta-cs)} - \frac{\lambda}{(\lambda+\beta+\delta-cs)} - \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right| |\hat{f}_2(s)| \to 0$$

on C. For $r \to \infty$, and hence it holds that

$$\Big| \left[\frac{\lambda}{(\lambda+\beta+\delta-cs)} + \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right] \hat{f}_1(s) + \left[\frac{\lambda}{(\lambda+\delta-cs)} - \frac{\lambda}{(\lambda+\beta+\delta-cs)} - \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right] \hat{f}_2(s) \Big| < 1. \tag{6}$$

on C.

(2) For $Re(s) = 0$ and for $\delta > 0$, similar to Cossette et al.(2008)[10], we let

$$\hat{d}_\delta(s) = \frac{\lambda}{\lambda+\delta-cs} - \frac{\lambda}{\lambda+\beta+\delta-cs} - \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2}$$

then we have

$$|\hat{d}_\delta(s)| = \left| \frac{\lambda}{\lambda+\delta-cs} - \frac{\lambda}{\lambda+\beta+\delta-cs} - \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right|$$
$$= \lambda \left| \frac{\beta^2}{(\lambda+\delta-cs)(\lambda+\beta+\delta-cs)^2} \right|$$
$$\leq \lambda \left| \frac{\beta^2}{(\lambda+\delta)(\lambda+\beta+\delta)} \right| = |\hat{d}_\delta(0)|$$

and

$$\Big| \left[\frac{\lambda}{(\lambda+\beta+\delta-cs)} + \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right] \hat{f}_1(s) + \left[\frac{\lambda}{(\lambda+\delta-cs)} - \frac{\lambda}{(\lambda+\beta+\delta-cs)} - \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right] \hat{f}_2(s) \Big|$$
$$= \left| \left(\frac{\lambda}{\lambda+\beta+\delta-cs} + \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right) \hat{f}_1(s) + \hat{f}_2(s)\hat{d}_\delta(s) \right|$$
$$\leq \left| \frac{\lambda}{\lambda+\beta+\delta-cs} \right| + \left| \frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2} \right| + |\hat{d}_\delta(s)|$$
$$\leq \left| \frac{\lambda}{\lambda+\beta+\delta} \right| + \left| \frac{\lambda\beta}{(\lambda+\beta+\delta)^2} \right| + |\hat{d}_\delta(0)|. \tag{7}$$

For $\delta > 0$, it holds $\hat{d}_\delta(0) > 0$. Indeed,

$$\hat{d}_\delta(0) = \frac{\lambda}{\lambda+\delta} - \frac{\lambda}{\lambda+\beta+\delta} - \frac{\lambda\beta}{(\lambda+\beta+\delta)^2} = \frac{\lambda\beta^2}{(\lambda+\beta+\delta)^2(\lambda+\beta)} > 0.$$

Therefore, for s on the imaginary axis and for $\delta > 0$, Equation (7) becomes

$$\begin{aligned}&\Bigg|\left[\frac{\lambda}{(\lambda+\beta+\delta-cs)}+\frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2}\right]\hat{f}_1(s)\\&+\left[\frac{\lambda}{(\lambda+\delta-cs)}-\frac{\lambda}{(\lambda+\beta+\delta-cs)}-\frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2}\right]\hat{f}_2(s)\Bigg|\\&\leq\left|\frac{\lambda}{\lambda+\beta+\delta}\right|+\left|\frac{\lambda\beta}{(\lambda+\beta+\delta)^2}\right|+|\hat{d}_\delta(0)|\\&\leq\frac{(\lambda+\beta+\delta)^2-\delta^2-2\beta\delta-\lambda\delta}{(\lambda+\beta+\delta)^2}<1.\end{aligned}$$

Above all, we proved that

$$\begin{aligned}&|\lambda(\lambda+\delta-cs)(\lambda+2\beta+\delta-cs)\hat{f}_1(s)+\lambda\beta^2\hat{f}_2(s)|\\&<|(\lambda+\delta-cs)(\lambda+\beta+\delta-cs)^2|\end{aligned}$$

in two case, and thus by Rouche's theorem, it follows that Equation(5) has the same number of roots as the equation $(\lambda+\delta-cs)(\lambda+\beta+\delta-cs)^2$ inside C_r. Since the latter equation has exactly 3 positive roots inside C_r, that is, Equation (4) has exactly 3 roots, say $\rho_1(\delta),\ldots,\rho_3(\delta)$ with positive real parts. Finally, we complete the proof by letting $r\to\infty$.

In the following, for simiplicity we write ρ_j for $\rho_j(\delta), j=1,2,3.$ when $\delta>0$.

REMARK. For $\delta=0$, the conditions to Rouche´'s theorem are not satisfied, since

$$\begin{aligned}&\Bigg|\left[\frac{\lambda}{(\lambda+\beta+\delta-cs)}+\frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2}\right]\hat{f}_1(s)\\&+\left[\frac{\lambda}{(\lambda+\delta-cs)}-\frac{\lambda}{(\lambda+\beta+\delta-cs)}-\frac{\lambda\beta}{(\lambda+\beta+\delta-cs)^2}\right]\hat{f}_2(s)\Bigg|\\&=\left|\frac{\lambda}{\lambda+\beta}+\frac{\lambda\beta}{(\lambda+\beta)^2}+\left[1-\frac{\lambda}{\lambda+\beta}-\frac{\lambda\beta}{\lambda+\beta}\right]\right|=1\end{aligned}$$

for $s=0$. The case of $\delta=0$ is important to evaluate ruin probablity, being special cases of the Gerber-Shiu penalty function at $\delta=0$. We apply the Klimenok(2001)[11] to derive the number of roots to the generalized Lundberg's equation with $\delta=0$.

PROPOSITION 2. For $\delta=0$, Lundberg's generalised Equation(4) has exaclty 2 roots, say $\rho_1(0)$, $\rho_2(0)$, with positive real parts and one root equals zero.

Proof. Define the contour $D_k = s : |z| = 1$ and let $z=1-\frac{s}{k}$. In terms of s, the contour D_k is a circle with origin at k and radius k. Similarly as in Proposition 1, we let $k\to\infty$ and denote by D the limiting contour.Using the same arguments as in the proof of Proposition 1, one can show that Equation(??) also holds on D (excluding s=0 or equivalently z=1) for $\delta=0$. besides, the functions $\lambda(\lambda+\delta-cs)(\lambda+2\beta+\delta-cs)\hat{f}_1(s)+\lambda\beta^2\hat{f}_2(s)$ and $(\lambda+\delta-cs)(\lambda+\beta+\delta-cs)^2$ are continuous on D. As Theorem 1 of Klimenok(2001), we need prove that

$$\begin{aligned}&\frac{d}{dz}\Bigg\{1-\left[\frac{\lambda}{(\lambda+\beta+\delta-ck(1-z))}+\frac{\lambda\beta}{(\lambda+\beta+\delta-ck(1-z))^2}\right]\hat{f}_1(k-kz)\\&-\left[\frac{\lambda}{(\lambda+\delta-ck(1-z))}-\frac{\lambda}{(\lambda+\beta+\delta-ck(1-z))}\right.\\&\left.-\frac{\lambda\beta}{(\lambda+\beta+\delta-ck(1-z))^2}\right]\hat{f}_2(k-kz)\Bigg\}\Bigg|_{z=1}>0.\end{aligned}$$

The left-hand side of this relation is equal to

$$\frac{d}{dz}\left\{1 - E\left[e^{(k-kz)(cW-X)}\right]\right\}\Bigg|_{z=1} = kE\left[cW - X\right]$$

where $E\left[cW - X\right] > 0$ given the solvability condition in equation (1).

Based on Klimenok(2001), we conclude the number of roots of Equation (5) is equal to 2 inside D, that is, the number of roots of $(\lambda + \delta - cs)(\lambda + \beta + \delta - cs)^2$ inside D minus 1. Moreover, a trival root to Lundberg's generalised equation (4) equals zero.

4 Laplace Transform of $m_\delta(u)$

In this section, we want to derive the LT $\hat{m}_\delta(s)$ of the Gerber-Shiu expected discount penalty function $m_\delta(u)$. For $u \geq 0$ and setting $y = u + ct$, we have

$$\begin{aligned}
m_\delta(u) &= E[e^{-\delta\tau} w(U_{\tau-}, |U_\tau|) 1_{\tau<\infty} | U_0 = u] \\
&= \frac{\lambda}{c}\int_u^\infty e^{-(\delta+\lambda+\beta)(\frac{y-u}{c})}(\sigma_1(y) - \sigma_2(y))dy \\
&+ \frac{\lambda}{c}\frac{1}{\beta}\int_u^\infty e^{-(\delta+\lambda)(\frac{y-u}{c})} f\left(\frac{y-u}{c}\right)(\sigma_1(y) - \sigma_2(y))dy \\
&+ \frac{\lambda}{c}\int_u^\infty e^{-(\delta+\lambda)(\frac{y-u}{c})}\sigma_2(y)dy,
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{1,\delta}(u) &= \int_0^u m_\delta(u-x) f_1(x)dx + \gamma_1(u), \quad \gamma_1(u) = \int_u^\infty f_1(x)dx, \\
\sigma_{2,\delta}(u) &= \int_0^u m_\delta(u-x) f_2(x)dx + \gamma_2(u), \quad \gamma_2(u) = \int_u^\infty f_2(x)dx.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\frac{c}{\lambda}\hat{m}_\delta(s) &= \int_0^\infty e^{-su}\frac{c}{\lambda} m_\delta(u)du \\
&= \int_0^\infty e^{-\frac{\lambda+\beta+\delta}{c}y}\left(\sigma_1(y) - \sigma_2(y)\right)\int_0^y e^{-\left(s-\frac{\lambda+\beta+\delta}{c}u\right)}dudy \\
&+ \frac{\beta}{c}\int_0^\infty e^{-\frac{\lambda+\beta+\delta}{c}y}\left(\sigma_1(y) - \sigma_2(y)\right)\int_0^y e^{-\left(s-\frac{\lambda+\beta+\delta}{c}u\right)}(y-u)dudy \\
&+ \int_0^\infty e^{-\frac{\lambda+\delta}{c}y}\sigma_2(y)\int_0^y e^{-\left(s-\frac{\lambda+\delta}{c}u\right)}dudy. \qquad (8)
\end{aligned}$$

It can be easily proved that for $a > 0$

$$\begin{aligned}
\int_0^y e^{-au}du &= -\frac{e^{-ay}}{a} \\
\int_0^y (y-u)e^{-au}du &= \frac{y}{a} - \frac{1}{a^2} + \frac{e^{-ay}}{a^2} \qquad (9)
\end{aligned}$$

Therefore, using Equation (9), Equation (8) can be written in the form

$$\begin{aligned}
\frac{c}{\lambda}\hat{m}_\delta(s) &= \frac{1}{\left(\frac{\lambda+\beta+\delta}{c} - s\right)}\left(\hat{\sigma}_1(s) - \hat{\sigma}_2(s)\right) + \frac{1}{\left(\frac{\lambda+\delta}{c} - s\right)}\hat{\sigma}_2(s) \\
&+ \frac{\beta}{c}\frac{1}{\left(s - \frac{\lambda+\beta+\delta}{c}\right)^2}\left(\hat{\sigma}_1(s) - \hat{\sigma}_2(s)\right) + \hat{B}_\delta(s). \qquad (10)
\end{aligned}$$

where

$$\hat{\sigma}_{i,\delta}(s) = \int_0^\infty e^{-su}\sigma_{i,\delta}(u)du \quad i = 1,2.$$

and

$$\hat{B}_\delta(s) = \int_0^\infty ye^{-(\delta+\lambda+\beta)\frac{y}{c}}(\sigma_1(y) - \sigma_2(y))\frac{1}{\left(s - \frac{\lambda+\delta+\beta}{c}\right)}dy$$
$$- \int_0^\infty e^{-(\delta+\lambda+\beta)\frac{y}{c}}(\sigma_1(y) - \sigma_2(y))\frac{1}{(s - \frac{\delta+\lambda+\beta}{c})^2}dy$$

Let $\hat{\gamma}_i(s) = \int_0^\infty e^{-su}\gamma_i(u)du, \quad i = 1,2$, the above Equation (10) reduces to

$$\hat{m}_\delta(s)\left[\frac{c}{\lambda} - \frac{\hat{f}_1(s) - \hat{f}_2(s)}{\left(\frac{\lambda+\beta+\delta}{c} - s\right)} - \frac{\hat{f}_2(s)}{\left(\frac{\lambda+\delta}{c} - s\right)} - \frac{\hat{f}_1(s) - \hat{f}_2(s)}{\frac{c}{\beta}\left(\frac{\lambda+\beta+\delta}{c} - s\right)^2}\right]$$
$$= \frac{\hat{\gamma}_1(s) - \hat{\gamma}_2(s)}{\left(\frac{\lambda+\beta+\delta}{c} - s\right)} + \frac{\hat{\gamma}_2(s)}{\left(\frac{\lambda+\delta}{c} - s\right)} + \frac{\hat{\gamma}_1(s) - \hat{\gamma}_2(s)}{\frac{c}{\beta}\left(\frac{\lambda+\beta+\delta}{c} - s\right)^2} + \hat{B}_\delta(s). \tag{11}$$

Now using Equation (11), we give the following theorem about the expression for $\hat{m}_\delta(s)$.

THEOREM 1. In this risk process with a dependence structure, the LT $\hat{m}_\delta(s)$ of the $m_\delta(u)$ is given by

$$\hat{m}_\delta(s) = \frac{\hat{\beta}_{1,\delta}(s) + \hat{\beta}_{2,\delta}(s)}{\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s)}, \tag{12}$$

where

$$\hat{h}_{1,\delta}(s) = \frac{c}{\lambda}\frac{c}{\beta}\left(\frac{\delta+\lambda+\beta}{c} - s\right)^2\left(\frac{\delta+\lambda}{c} - s\right), \tag{13}$$

$$\hat{h}_{2,\delta}(s) = \left(\frac{\delta+\lambda}{c} - s\right)\left[\frac{c}{\beta}\left(\frac{\delta+\lambda+\beta}{c} - s\right) + 1\right]\hat{f}_1(s) + \frac{\beta}{c}\hat{f}_2(s), \tag{14}$$

$$\hat{\beta}_{1,\delta}(s) = \left(\frac{\delta+\lambda}{c} - s\right)\left[\frac{c}{\beta}\left(\frac{\delta+\lambda+\beta}{c} - s\right) + 1\right]\hat{\gamma}_1(s) + \frac{\beta}{c}\hat{\gamma}_2(s), \tag{15}$$

$$\hat{\beta}_{2,\delta}(s) = -\sum_{j=1}^{3}\hat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{3}\frac{s - \rho_k}{\rho_j - \rho_k}.$$

Proof. Multiplying both sides of Equation (11) by $\frac{c}{\beta}\left(\frac{\delta+\lambda+\beta}{c} - s\right)^2\left(\frac{\delta+\lambda}{c} - s\right)$ and then solving the resulting equation for $\hat{m}_\delta(s)$ we get immediately the equation (12), with

$$\hat{\beta}_{2,\delta}(s) = \left(\frac{\lambda+\delta}{c} - s\right)\left(s - \frac{\lambda+\delta+\beta}{c}\right)^2\hat{B}_\delta(s)$$
$$= \left(\frac{\lambda+\delta}{c} - s\right)\left(s - \frac{\lambda+\delta+\beta}{c}\right)^2\left[\int_0^\infty ye^{-(\delta+\lambda+\beta)\frac{y}{c}}(\sigma_1(y) - \sigma_2(y))\frac{1}{\left(s - \frac{\lambda+\delta+\beta}{c}\right)}dy\right.$$
$$\left.- \int_0^\infty e^{-(\delta+\lambda+\beta)\frac{y}{c}}(\sigma_1(y) - \sigma_2(y))\frac{1}{(s - \frac{\delta+\lambda+\beta}{c})^2}dy\right]$$
$$= \left(\frac{\lambda+\delta}{c} - s\right)\left(\frac{\lambda+\delta+\beta}{c} - s\right)\hat{\mu}_1\left(\frac{\lambda+\beta+\delta}{c}\right) + \left(\frac{\lambda+\delta}{c} - s\right)\hat{\mu}_0\left(\frac{\lambda+\beta+\delta}{c}\right).$$

which is a polynomial in s of degree 3 or less, where

$$\hat{\mu}_j\left(\frac{\delta+\lambda+\beta}{c}\right) = \int_0^\infty e^{-(\delta+\lambda+\beta)y/c}(\sigma_1(y) - \sigma_2(y))y^j dy \quad (j = 0,1).$$

The Lundberg's generalised equation (4) can be written as $\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) = 0$, which means that $\rho_i' s, i = 1, \ldots, 3$ are roots of the denominator in Equation (12). Since $\hat{m}_\delta(s)$ is analytic for $Re(s) \geq 0$, it means that $\rho_i' s, i = 1, \ldots, 3$ are also roots of the numerator in Equation (12), and thus $\hat{\beta}_{2,\delta}(\rho_i) = -\hat{\beta}_{1,\delta}(\rho_i), i = 1, \ldots, 3$. Since $\hat{\beta}_{2,\delta}(s)$ is a polynomial in s of degree 2, by using Lagrange interpolation formula at the 3 points ρ_1, ρ_2, ρ_3, we have

$$\hat{\beta}_{2,\delta}(s) = \sum_{j=1}^{3} \hat{\beta}_{2,\delta}(\rho_j) \prod_{k=1,k\neq j}^{3} \frac{s-\rho_k}{\rho_j - \rho_k} = -\sum_{j=1}^{3} \hat{\beta}_{1,\delta}(\rho_j) \prod_{k=1,k\neq j}^{3} \frac{s-\rho_k}{\rho_j - \rho_k},$$

and then the proof is completed.

5 The Gerber-Shiu penalty function when u=0

In this section, we look at the Gerber-Shiu penalty function $m_\delta(0)$, the LT of the time of ruin $m_\tau(0)$ and the ruin probability when u=0.

THEOREM 2. When u=0, the Gerber-Shiu penalty function $m_\delta(0)$:

$$m_\delta(0) = \sum_{j=1}^{3} \frac{\hat{\beta}_{1,\delta}(\rho_j)}{\prod_{k=1,k\neq j}^{3}(\rho_k - \rho_j)}.$$

Proof. We assume that the roots of Lundberg's equation ρ_1, ρ_2, ρ_3 are all distinct. By applying the initial value theorem, we get

$$\begin{aligned} m_\delta(0) &= \lim_{s\to\infty} s\hat{m}(s) = \lim_{s\to\infty} s\frac{\hat{\beta}_{1,\delta}(s) + \hat{\beta}_{2,\delta}(s)}{\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s)} \\ &= \lim_{s\to\infty} s\frac{\hat{\beta}_{1,\delta}(s) - \sum_{j=1}^{3} \hat{\beta}_{1,\delta}(\rho_j) \prod_{k=1,k\neq j}^{3} \frac{s-\rho_k}{\rho_j-\rho_k}}{\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s)} \\ &= \lim_{s\to\infty} \frac{\frac{\hat{\beta}_1(s)}{s^3} - \frac{1}{s^3}\sum_{j=1}^{3} \hat{\beta}_{1,\delta}(\rho_j) \prod_{k=1,k\neq j}^{3} \frac{s-\rho_k}{\rho_j-\rho_k}}{\frac{\hat{h}_1(s)}{s^4} - \frac{\hat{h}_2(s)}{s^4}} \\ &= \lim_{s\to\infty} \frac{-\frac{1}{s^2}\sum_{j=1}^{3} \hat{\beta}_{1,\delta}(\rho_j) \prod_{k=1,k\neq j}^{3} \frac{s-\rho_k}{\rho_j-\rho_k}}{(-1)^3} \\ &= \sum_{j=1}^{3} \frac{\hat{\beta}_{1,\delta}(\rho_j)}{\prod_{k=1,k\neq j}^{3}(\rho_k - \rho_j)}. \end{aligned} \tag{16}$$

THEOREM 3. When u=0, the Laplace Transform of the time ruin $m_\tau(0)$ is:

$$m_\tau(0) = 1 - \frac{(\delta+\lambda+\beta)^2\delta}{c\lambda\beta\prod_{i=1}^{3}\rho_i}.$$

Proof. Let

$$b_{1,\delta}(s) = \left(\frac{\lambda+\delta}{c} - s\right)\left[\frac{c}{\beta}\left(\frac{\lambda+\beta+\delta}{c} - s\right) + 1\right], \tag{17}$$

$$b_{2,\delta}(s) = \frac{\beta}{c}. \tag{18}$$

And from Equation (15) we have

$$\hat{\beta}_{1,\delta}(s) = b_{1,\delta}(s)\hat{\gamma}_1(s) + b_{2,\delta}(s)\hat{\gamma}_2(s). \tag{19}$$

then $m_\delta(0)$ becomes

$$m_\delta(0) = \sum_{j=1}^{3} \frac{b_{1,\delta}(\rho_j)\hat{\gamma}_1(\rho_j) + b_{2,\delta}(\rho_j)\hat{\gamma}_2(\rho_j)}{\prod_{k=1,k\neq j}^{3}(\rho_k - \rho_j)} = \sum_{i=1}^{2}\sum_{j=1}^{3} b_{i,j}\hat{\gamma}_i(\rho_j), \tag{20}$$

which

$$b_{i,j} = \frac{b_{i,\delta}(\rho_j)}{\prod_{k=1,k\neq j}^{3}(\rho_k - \rho_j)}, i = 1, 2. j = 1, 2, 3. \tag{21}$$

Since

$$\hat{\gamma}_i(s) = \int_0^\infty e^{-sx}\gamma_i(x)dx = \int_0^\infty \int_0^\infty e^{-sx} w(x,y) f_i(x+y) dy dx,$$

then we have

$$m_\delta(0) = \int_0^\infty \int_0^\infty w(x,y) \left[f_1(x+y)\sum_{j=1}^{3} b_{1,j}e^{-\rho_j x} + f_2(x+y)\sum_{j=1}^{3} b_{2,j}e^{-\rho_j x} \right] dy dx. \tag{22}$$

Let $f(x,y,t|0)$ be the joint defective density of the surplus prior to ruin (x), the deficit at ruin (y) and the time of ruin (t) given U(0)=0, and $f_\delta(x,y|0)$ be the discounted (nondiscounted if $\delta \to 0$) p.f.d. of the surplus just before ruin and the deficit at ruin. The relationship between the two is

$$f_\delta(x,y|0) = \int_0^\infty e^{-\delta t} f(x,y,t|0)dt.$$

For u=0, and from Equation(16) of Cheung et al.(2010)[12], it obtains that

$$m_\delta(0) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta t} w(x,y) f(x,y,t|0) dt dy dx = \int_0^\infty \int_0^\infty w(x,y) f_\delta(x,y|0) dy dx,$$

which combined with Equation (22) yields

$$f_\delta(x,y|0) = f_1(x+y)\sum_{j=1}^{3} b_{1,j}e^{-\rho_j x} + f_2(x+y)\sum_{j=1}^{3} b_{2,j}e^{-\rho_j x}. \tag{23}$$

We also let $f_{1,\delta}(x|0) = \int_0^\infty f_\delta(x,y|0)dy$ be the discounted p.d.f. of the surplus prior to ruin and $f_{2,\delta}(y|0) = \int_0^\infty f_\delta(x,y|0)dx$ be the discounted p.d.f. of the deficit at ruin given U(0)=0. Since

$$\int_0^\infty f_i(x+y)dy = \int_x^\infty f_i(y)dy = 1 - F_i(x) = \bar{F}_i(x), i = 1, 2.$$

From Equation (??) we obtain

$$f_{1,\delta}(x|0) = \int_0^\infty f_\delta(x,y|0)dy = \bar{F}_1(x)\sum_{j=1}^{3} b_{1,j}e^{-\rho_j x} + \bar{F}_2(x)\sum_{j=1}^{3} b_{2,j}e^{-\rho_j x},$$

and using the Property of the Dickson-Hipp operator on page 394 of Li and Garrido(2004)[13], we have

$$\begin{aligned} f_{2,\delta}(y|0) &= \int_0^\infty f_\delta(x,y|0)dx \\ &= \int_0^\infty \left(f_1(x+y)\sum_{j=1}^{3} b_{1,j}e^{-\rho_j x} + f_2(x+y)\sum_{j=1}^{3} b_{2,j}e^{-\rho_j x} \right) dx \\ &= \sum_{j=1}^{3} b_{1,j}T_{\rho_j}f_1(y) + \sum_{j=1}^{3} b_{2,j}T_{\rho_j}f_2(y). \end{aligned} \tag{24}$$

The Laplace transform of $f_{2,\delta}(y|0)$ is that

$$\begin{aligned}\hat{f}_{2,\delta}(s) &= \int_0^\infty e^{-sy} f_{2,\delta}(y|0)dy = T_s f_{2,\delta}(0|0) \\ &= \sum_{j=1}^{3} b_{1,j} T_s T_{\rho_j} f_1(0) + \sum_{j=1}^{3} b_{2,j} T_s T_{\rho_j} f_2(0) \\ &= \sum_{j=1}^{3} \frac{b_{1,\delta}\hat{f}_1(\rho_j) + b_{2,j}\hat{f}_2(\rho_j)}{s-\rho_j} - \hat{f}_1(s)\sum_{j=1}^{3}\frac{b_{1,j}}{s-\rho_j} - \hat{f}_2(s)\sum_{j=1}^{3}\frac{b_{2,j}}{s-\rho_j}. \end{aligned} \tag{25}$$

Using Equation (22), (17) and (18), it follows that $\hat{h}_{2,\delta}(s) = b_{1,\delta}(s)\hat{f}_1(s) + b_{2,\delta}(s)\hat{f}_2(s)$, and thus for $j = 1, \ldots, 3$, it holds

$$\begin{aligned} b_{1,j}\hat{f}_1(\rho_j) + b_{2,j}\hat{f}_2(\rho_j) &= \frac{b_{1,\delta}(\rho_j)\hat{f}_1(\rho_j) + b_{2,\delta}(\rho_j)\hat{f}_2(\rho_j)}{\prod_{k=1,k\neq j}^{3}(\rho_k-\rho_j)} = \frac{\hat{h}_{2,\delta}(\rho_j)}{\prod_{k=1,k\neq j}^{3}(\rho_k-\rho_j)} \\ &= \frac{\hat{h}_{1,\delta}(\rho_j)}{\prod_{k=1,k\neq j}^{3}(\rho_k-\rho_j)}. \end{aligned}$$

Then using Equation (22) and (25), we have

$$\begin{aligned}\hat{f}_{2,\delta}(s) &= \sum_{j=1}^{3}\frac{(\delta+\lambda+\beta-c\rho_j)^2(\delta+\lambda-c\rho_j)}{c\lambda\beta(s-\rho_j)\prod_{k=1,k\neq j}^{3}(\rho_k-\rho_j)} - \hat{f}_1(s)\sum_{j=1}^{3}\frac{b_{1,j}}{s-\rho_j} \\ &\quad - \hat{f}_2(s)\sum_{j=1}^{3}\frac{b_{2,j}}{s-\rho_j}. \end{aligned} \tag{26}$$

According to the similar argument from interpolation theroy as in Li and Garrido(2005)[14] of the Equation(17)and (18), Equation (26) rewrites as

$$\begin{aligned}\hat{f}_{2,\delta}(s) &= 1 - \frac{(\delta+\lambda+\beta-cs)^2(\delta+\lambda-cs)}{c\lambda\beta\prod_{i=1}^{3}(\rho_i-s)} \\ &\quad + \hat{f}_1(s)\frac{(\lambda+\delta-cs)(\lambda+\delta+\beta-cs+\beta)}{c\beta\prod_{i=1}^{3}(\rho_i-s)} + \hat{f}_2(s)\frac{\beta}{c\prod_{i=1}^{3}(\rho_i-s)} \\ &= 1 - \frac{1}{\prod_{i=1}^{3}(\rho_i-s)}\left\{\frac{c^2}{\lambda\beta}\left(\frac{\delta+\lambda}{c}-s\right)\left(\frac{\lambda+\beta+\delta}{c}-s\right)^2\right. \\ &\quad \left. -\left(\frac{\lambda+\delta}{c}-s\right)\left[\left(\frac{\lambda+\beta+\delta}{c}-s\right)\frac{c}{\beta}+1\right]\hat{f}_1(s) - \frac{\beta}{c}\hat{f}_2(s)\right\} \\ &= 1 - \frac{\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)}{\prod_{i=1}^{3}(\rho_i-s)}. \end{aligned}$$

Let $w(x,y) = 1$, when $U(0) = 0$, since $\hat{f}_1(0) = 1, \hat{f}_2(0) = 1$, the LT of the time of ruin m_τ given U(0)=0, is

$$\begin{aligned} m_\tau(0) &= E\left[e^{-\delta\tau}I(\tau<\infty)|U(0)=0\right] = \int_0^\infty\int_0^\infty f_\delta(u,y|0)dydu \\ &= \int_0^\infty f_{2,\delta}(y|0)dy = \lim_{s\to 0}\hat{f}_{2,\delta}(s) = 1 - \frac{\hat{h}_{1,\delta}(0)-\hat{h}_{2,\delta}(0)}{\rho_1\rho_2\rho_3} \\ &= 1 - \frac{(\delta+\lambda+\beta)^2\delta}{c\lambda\beta\prod_{i=1}^{3}\rho_i}. \end{aligned} \tag{27}$$

Due to $\delta > 0$, we can derive that $m_\tau(0) < 1$.

THEOREM 4. When the initial surplus U(0)=0, the ruin probability $\psi(0)$ is

$$\psi(0) = 1 - \frac{(\lambda+\beta)^2}{c\lambda\beta\rho_1^{'}(0)\rho^*(0)}.$$

Proof. When U(0)=0,

$$\begin{aligned}\psi(0) &= \lim_{\delta\to 0^+} E\left[e^{-\delta\tau}I(\tau<\infty)|U(0)=u\right] \\ &= 1 - \lim_{\delta\to 0^+}\frac{(\delta+\lambda+\beta)^2\delta}{c\lambda\beta\prod_{i=1}^{3}\rho_i} \\ &= 1 - \frac{(\lambda+\beta)^2}{c\lambda\beta\rho_1^{'}(0)\rho^*(0)},\end{aligned} \tag{28}$$

where $\rho^*(0) = \prod_{i=2}^{3}\rho_i(0)$ and $\rho_1^{'}(0) = \frac{d}{d\delta}\rho_1(\delta)|_\delta \to 0^+$. Using the fact that $\rho_1(\delta)$ is a root of Lundberg equation, we have $\hat{h}_1(\rho_1(\delta)) = \hat{h}_2(\rho_1(\delta))$. By differentiating with respect to δ and then letting $\delta \to 0^+$, we obtain

$$(\lambda+\beta)^2(1-c\rho_1^{'}(0)) = -\lambda^2(\lambda+2\beta)\mu_1\rho_1^{'}(0) - \lambda\beta^2\mu_2\rho_1^{'}(0) \tag{29}$$

where $\hat{f}_1^{'}(0) = -\mu_1, \hat{f}_2^{'}(0) = -\mu_2$. From Equation (29), we have that

$$\rho_1^{'}(0) = \frac{(\lambda+\beta)^2}{c(\lambda+\beta)^2 - \lambda^2(\lambda+2\beta)\mu_1 - \lambda\beta^2\mu_2} = \frac{E(W)}{cE(W)-E(X)} \tag{30}$$

which is always positive due to the positive loading condition (see Equation (1)). Therefore, using Equation (28) and (30), we have that

$$\psi(0) = 1 - \frac{(\lambda+\beta)^2}{c\lambda\beta\rho_1^{'}(0)\rho^*(0)} = 1 - \frac{[cE(W)-E(X)]}{c\lambda\beta\rho^*(0)} < 1. \tag{31}$$

6 Defection renewal function

THEOREM 5. According to the Gerber-Shiu penalty function $m_\delta(u)$, we can get that

$$m_\delta(u) = \int_0^u m_\delta(u-y)\zeta_\delta(y)dy + G_\delta(u), \quad u \ge 0. \tag{32}$$

where

$$\begin{aligned}\zeta_\delta(y) &= T_{\rho_1}T_{\rho_2}T_{\rho_3}h_{2,\delta}(u), \\ G_\delta(u) &= T_{\rho_1}T_{\rho_2}T_{\rho_3}\beta_{1,\delta}(u).\end{aligned} \tag{33}$$

Proof. Since $\int_0^\infty f_{2,\delta}(y|0)dy = m_\tau(0) < 1$ (from Equation (27)), Equation (32) is a defective renewal equation. Using the Lagrange interpolating formula, we derive that

$$\hat{h}_{1,\delta}(s) = \hat{h}_{1,\delta}(0)\prod_{k=1}^{3}\frac{s-\rho_k}{(-\rho_k)} + s\sum_{j=1}^{3}\frac{\hat{h}_{1,\delta}(\rho_j)}{\rho_j}\prod_{k=1,k\neq j}^{3}\frac{s-\rho_k}{\rho_j-\rho_k}.$$

Similar arguments as the Cossette et al.(2010)[5], the aforementioned relation implies that

$$\begin{aligned}\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) = \pi_3(s)\Big[&\frac{\hat{h}_{1,\delta}(0)}{\pi_3(0)} - \sum_{j=1}^{3}\frac{\hat{h}_{2,\delta}(\rho_j)}{(-\rho_j)\pi_3^{'}(\rho_j)} \\ &+ \sum_{j=1}^{3}\frac{\hat{h}_{2,\delta}(\rho_j)}{(s-\rho_j)\pi_3^{'}(\rho_j)} - \frac{\hat{h}_{2,\delta}(s)}{\pi_3(s)}\Big],\end{aligned} \tag{34}$$

where $\pi_3(s)=\prod_{i=1}^{3}(s-\rho_j)$. Since $\hat{h}_{2,\delta}(\rho_j)=\hat{h}_{1,\delta}(\rho_j)$, $j=1,2,3$, for s=0, we obtain

$$\begin{aligned}\frac{\hat{h}_{1,\delta}(0)}{\pi(0)}+\sum_{j=1}^{3}\frac{\hat{h}_{2,\delta}(\rho_j)}{\rho_j\pi'(\rho_j)} &= \frac{\frac{c^2}{\lambda\beta}\left(\frac{\delta+\lambda+\beta}{c}\right)^2\left(\frac{\delta+\lambda}{c}\right)}{\prod_{i=1}^{3}(-\rho_i)}+\sum_{j=1}^{3}\frac{\frac{c^2}{\lambda\beta}\left(\frac{\delta+\lambda}{c}-\rho_j\right)\left(\frac{\lambda+\delta+\beta}{c}-\rho_j\right)^2}{\rho_j\prod_{k=1,k\neq j}^{3}(\rho_j-\rho_k)}\\ &= \frac{(\delta+\lambda+\beta)^2(\delta+\lambda)}{c\lambda\beta\prod_{i=1}^{3}(-\rho_j)}+(-1)^3\left[1-\frac{(\delta+\lambda+\beta)^2(\delta+\lambda)}{c\lambda\beta\prod_{i=1}^{3}(\rho_j)}\right]\\ &= -1.\end{aligned}$$

Then Equation (34) becomes

$$\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)=(-1)^3\pi_3(s)\left[1-T_sT_{\rho_1}T_{\rho_2}T_{\rho_3}h_{2,\delta}(0)\right]. \tag{35}$$

Furthermore, from (35) we get that

$$\begin{aligned}\hat{f}_{2,\delta}(s) &= 1-\frac{\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)}{\prod_{i=1}^{3}(\rho_i-s)}\\ &= 1-\frac{(-1)^3\pi_3(s)\left[1-T_sT_{\rho_1}T_{\rho_2}T_{\rho_3}h_{2,\delta}(0)\right]}{(-1)^3\pi_3(s)}\\ &= T_sT_{\rho_1}T_{\rho_2}T_{\rho_3}h_{2,\delta}(0). \end{aligned}\tag{36}$$

Since $\hat{f}_{2,\delta}(s)=\int_0^\infty e^{-su}f_{2,\delta}(y|0)du$, then

$$f_{2,\delta}(y|0)=T_{\rho_1}T_{\rho_2}T_{\rho_3}h_{2,\delta}(u).$$

Using the Dickson-Hipp operator, we have

$$\begin{aligned}G_\delta(u) &= \int_0^\infty\int_u^\infty w(s,t)\left[f_1(s+t)\sum_{j=1}^{3}b_{1,j}e^{-\rho_j(s-u)}+f_2(s+t)\sum_{j=1}^{3}b_{2,j}e^{-\rho_j(s-u)}\right]dsdt\\ &= \sum_{j=1}^{3}b_{1,j}\int_u^\infty e^{-\rho_j(s-u)}\gamma_1(s)ds+\sum_{j=1}^{3}b_{2,j}\int_u^\infty e^{-\rho_j(s-u)}\gamma_2(s)ds\\ &= \sum_{i=1}^{2}\sum_{j=1}^{3}b_{ij}T_{\rho_j}\gamma_i(u). \end{aligned}\tag{37}$$

From Equation (37), we obtain that the LT of the $G_\delta(u)$,

$$\begin{aligned}\hat{G}_\delta(s) &= \int_0^\infty e^{-su}G_\delta(u)du=T_sG_\delta(0)=\sum_{i=1}^{2}\sum_{j=1}^{3}b_{ij}T_sT_{\rho_j}\gamma_i(0)\\ &= \sum_{j=1}^{3}\frac{b_{1,\delta}\hat{\gamma}_1(\rho_j)+b_{2,\delta}\hat{\gamma}_2(\rho_j)}{(s-\rho_j)}-\hat{\gamma}_1(s)\sum_{j=1}^{3}\frac{b_{1,j}}{s-\rho_j}-\hat{\gamma}_2(s)\sum_{j=1}^{3}\frac{b_{2,j}}{s-\rho_j}\\ &= (-1)^3\left[\frac{\hat{\beta}_{1,\delta}(s)}{\pi(s)}-\sum_{j=1}^{3}\frac{\hat{\beta}_{1,\delta}(\rho_j)}{(s-\rho_j)\pi'(\rho_j)}\right]\\ &= T_sT_{\rho_1}T_{\rho_2}T_{\rho_3}\beta_{1,\delta}(0),\end{aligned}$$

Thus, by inverting Equation (37) we also get the alternative expression for $G_\delta(u)$,

$$G_\delta(u)=T_{\rho_1}T_{\rho_2}T_{\rho_3}\beta_{1,\delta}(u).$$

THEOREM 6. The defective renewal equation of $m_\tau(u)$ is:

$$m_\tau(u)=\int_0^u m_\tau(u-y)\zeta_\delta(y)dy+\int_u^\infty\zeta_\delta(y)dy,\quad u\geq 0. \tag{38}$$

7 Numberical illustration

In this section, we give some examples. If T_i is smaller than M_i, then the following claim size X_i has density function $f_1(x)$, otherwise its density function is $f_2(x)$. They are both exponential distribution with parameter λ_1, λ_2, that is, $f_1(x) = \lambda_1 e^{-\lambda_1 x}, f_2(x) = \lambda_2 e^{-\lambda_1 x}$, and $\hat{f}_1(s) = \frac{\lambda_1}{\lambda_1+s}, \hat{f}_2(s) = \frac{\lambda_2}{\lambda_2+s}$. We get an explicit expression for Taking LTs in both sides of the first equation in Theorem 6,

$$\hat{m}_\tau(s) = \frac{m_\tau(0) - \hat{\zeta}_{2,\delta}(s)}{s\left[1 - \hat{\zeta}_{2,\delta}(s)\right]} = \frac{1 - \hat{\zeta}_{2,\delta}(s) - [1 - m_\tau(0)]}{s\left[1 - \hat{\zeta}_{2,\delta}(s)\right]}. \tag{39}$$

From Equation (35) and (36) we have

$$\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) = \left[1 - \hat{\zeta}_{2,\delta}(s)\right] \prod_{i=1}^{3} (\rho_i - s),$$

and thus Equation (39) becomes

$$\hat{m}_\tau(s) = \frac{\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) - [1 - m_\tau(0)] \prod_{i=1}^{3} (\rho_i - s)}{s\left[\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s)\right]}. \tag{40}$$

From Equation (21),(22) we easily have

$$\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) = \frac{Q_{3,\delta}(s)}{c\lambda\beta(\lambda_1 + s)(\lambda_2 + s)}, \tag{41}$$

where

$$\begin{aligned} Q_{3,\delta}(s) = {} & (\lambda_1 + s)(\lambda_2 + s)(\lambda + \delta - cs)(\delta + \lambda + \beta - cs)^2 - \lambda\beta^2\lambda_2(\lambda_1 + s) \\ & - \lambda_1(\lambda_2 + s)(\lambda + \delta - cs)\left[\lambda(\lambda + \delta + \beta - cs) + \lambda\beta\right]. \end{aligned}$$

Since $Q_{3,\delta}(s)$ is a polynomial of degree 3 and then we have that $Q_{3,\delta}(s) = 0$ has 3 roots in the complex plane. Since $\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) = 0$ is Lundberg's generalised equation, that is equation $Q_{3,\delta}(s) = 0$ has 3 roots ρ_1, ρ_2, ρ_3, with positive real part and two roots say $-R_i = -R_i(\delta)$, with $Re(R_i) > 0, i = 1, 2$. Therefore, we can rewrite $Q_{3,\delta}(s)$ as

$$Q_{3,\delta}(s) = c\lambda\beta(s + R_1)(s + R_2) \prod_{i=1}^{3} (\rho_i - s). \tag{42}$$

So, from Equation (42)and (41), Equation (39) yields

$$\hat{m}_\tau(s) = \frac{\prod_{j=1}^{2} (s + R_j) - [1 - m_\tau(0)] (\lambda_1 + s)(\lambda_2 + s)}{s \prod_{j=1}^{2} (s + R_j)}. \tag{43}$$

Since $\hat{m}_\tau(s) < \infty$ for $s \geq 0$, the numerator in Equation (43) is zero for $s = 0$, that is

$$1 - m_\tau(0) = \frac{R_1 R_2}{\lambda_1 \lambda_2}$$

and then Equation (43) becomes

$$\hat{m}_\tau(s) = \frac{\left(1 - \frac{R_1 R_2}{\lambda_1 \lambda_2}\right) s + R_1 + R_2 - \frac{R_1 R_2 (\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2}}{(s + R_1)(s + R_2)}.$$

We assume that R_1, R_2 are distinct and use partial fractions yields

$$\hat{m}_\tau(s) = \sum_{j=1}^{2} \frac{\xi_{i,\delta}}{s + R_j},$$

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where

$$\xi_{1,\delta} = \frac{R_2}{R_2 - R_1}\left(1 - \frac{R_1(\lambda_1+\lambda_2)}{\lambda_1\lambda_2} + \frac{R_1^2}{\lambda_1\lambda_2}\right)$$

$$\xi_{2,\delta} = \frac{R_1}{R_2 - R_1}\left(1 - \frac{R_2(\lambda_1+\lambda_2)}{\lambda_1\lambda_2} + \frac{R_2^2}{\lambda_1\lambda_2}\right)$$

Inverting $\hat{m}_\tau(s)$ gives that

$$m_\tau(u) = \xi_{1,\delta}e^{-R_1 u} + \xi_{2,\delta}e^{-R_2 u}, u \geq 0, \tag{44}$$

and the ruin probability $\psi(u)$ can be obtained by letting $\delta \to 0$.

7.1 When $\delta = 0$

Let $\lambda_1 = 2$, $\lambda_2 = 4$, $c = 1.5$, $\lambda = 2$,
with $\beta = 1$,

$$\psi(u) = -0.00054031652280488e^{-3.9900610193824644u} + 0.6425490681568e^{-0.7155993125651344u},$$

with $\beta = 3$,

$$\psi(u) = -0.0041727320835225e^{-3.9383419808843664u} + 0.57252489270819e^{-0.8598590518898737u},$$

with $\beta = 5$,

$$\psi(u) = -0.00997224149241e^{-3.8733037753948447u} + 0.5182533288617954e^{-0.9744124540765244u},$$

with $\beta = 10$,

$$\psi(u) = -0.027705461665553e^{-3.7169851159041327u} + 0.43132361981897e^{-1.1643219472701969u}.$$

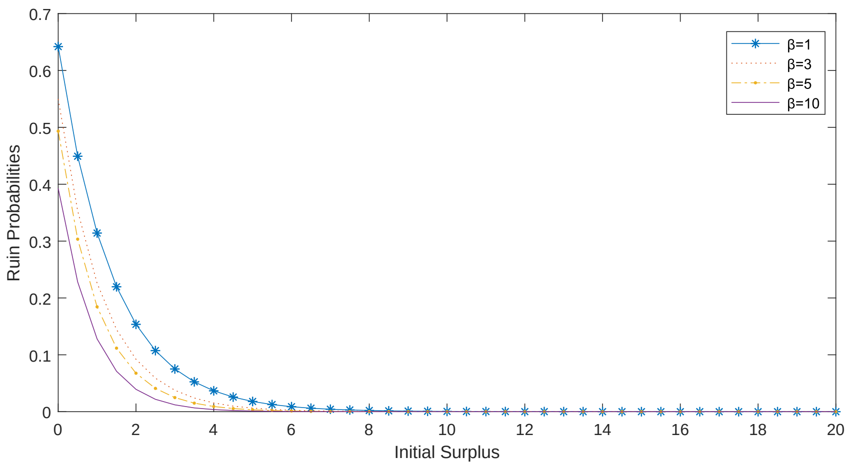

Figure 1: Ruin probabilities when $\delta = 0$

As we can see from Figure 1, the parameter β has a clear impact on the ruin probabilities.It is clear that the higher the parameter the lower the ruin probability is.

7.2 When $\delta = 1$

Furthermore using $\delta = 1$, we provide the analytic expressions for the LT of the time of ruin $m_\delta(u)$ as function of the initial surplus u, $(u \geq 0)$ and let $\lambda_1 = 2$, $\lambda_2 = 4$, $c = 1.5$, $\lambda = 2$, with $\beta = 1$,

$$\psi(u) = -0.000753258169851e^{-3.992695543278276u} + 0.4158061959884069e^{-1.16901584810085u},$$

with $\beta = 3$,

$$\psi(u) = -0.005125813528442236e^{-3.9533678066596094u} + 0.3884081526472388e^{-1.2272392825230218u},$$

with $\beta = 5$,

$$\psi(u) = -0.011401900767716385e^{-3.902448624640329u} + 0.3612870826739349e^{-1.2859844203405293u},$$

with $\beta = 10$,

$$\psi(u) = -0.02929701439042134e^{-3.7763937788329516u} + 0.30918222107489424e^{-1.401380900990933u}.$$

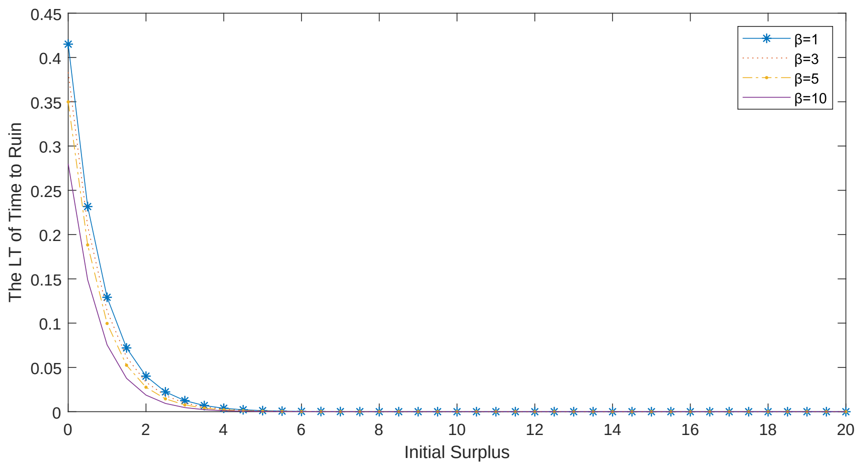

Figure 2: Ruin probabilities when $\delta = 1$

As we can see from Figure 2, the parameter β has a clear impact on the values of the LT of time to ruin. It is clear that the higher the parameter the lower the value of the LT of time to ruin is.

8 Conclusion

In this paper, we consider a risk model of claim amount affected by threshold. We derived the roots of the generalised Lundberg equation and the Laplace Transform of the expected discounted penalty function. Besides, the Gerber-Shiu penalty function is given when the initial surplus is zero and when it satifies some defective renewal equations. Some explicit expressions about the ruin probability are given to show that as the dependence parameter β is higer, the ruin probability and the value of the LT of time to ruin are both lower.

UNDER PEER REVIEW

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